Review : Power Series

Before looking at series solutions to a differential equation we will first need to do a cursory review of power series. A power series is a series in the form,

$$f(x)=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \tag{1}$$

where, x_0 and a_n are numbers. We can see from this that a power series is a function of x. The function notation is not always included, but sometimes it is so we put it into the definition above.

Before proceeding with our review we should probably first recall just what series really are. Recall that series are really just summations. One way to write our power series is then,

$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \\ &=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots \end{aligned} \tag{2}$$

Notice as well that if we needed to for some reason we could always write the power series as,

$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \\ &=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots \\ &=a_0+\sum_{n=1}^{\infty}a_n\left(x-x_0\right)^n \end{aligned}$$

All that we're doing here is noticing that if we ignore the first term (corresponding to $n=0$) the remainder is just a series that starts at $n=1$. When we do this we say that we've stripped out the $n=0$, or first, term. We don't need to stop at the first term either. If we strip out the first three terms we'll get,

$$\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n=a_0+a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+\sum_{n=3}^{\infty}a_n\left(x-x_0\right)^n$$

There are times when we'll want to do this so make sure that you can do it.

Now, since power series are functions of x and we know that not every series will in fact exist, it then makes sense to ask if a power series will exist for all x. This question is answered by looking at the convergence of the power series. We say that a power series **converges** for $x=c$ if the series,

$$\sum_{n=0}^{\infty}a_n\left(c-x_0\right)^n$$

converges. Recall that this series will converge if the limit of partial sums,

$$\lim_{N\to\infty}\sum_{n=0}^{N}a_n\left(c-x_0\right)^n$$

exists and is finite. In other words, a power series will converge for $x=c$ if

$$\sum_{n=0}^{\infty} a_n \left(c - x_0\right)^n$$

is a finite number.

Note that a power series will always converge if $x=x_0$. In this case the power series will become

$$\sum_{n=0}^{\infty} a_n \left(x_0 - x_0\right)^n = a_0$$

With this we now know that power series are guaranteed to exist for at least one value of x. We have the following fact about the convergence of a power series.

Fact

Given a power series, (1), there will exist a number $0 \le \rho \le \infty$ so that the power series will converge for $\left|x - x_0\right| < \rho$ and diverge for $\left|x - x_0\right| > \rho$. This number is called the **radius of convergence**.

Determining the radius of convergence for a power series is usually quite simple if we use the ratio test.

Ratio Test

Given a power series compute,

$$L = \left|x - x_0\right| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

then,

$$\begin{array}{lcl} L < 1 & \Rightarrow & \text{the series converges} \\ L > 1 & \Rightarrow & \text{the series diverges} \\ L = 1 & \Rightarrow & \text{the series may converge or diverge} \end{array}$$

Let's take a quick look at how this can be used to determine the radius of convergence.

Example 1 Determine the radius of convergence for the following power series.

$$\sum_{n=0}^{\infty} \frac{\left(-3\right)^n}{n\,7^{n+1}} \left(x - 5\right)^n$$

Solution

So, in this case we have,

$$a_n = \frac{\left(-3\right)^n}{n\,7^{n+1}} \qquad\qquad a_{n+1} = \frac{\left(-3\right)^{n+1}}{\left(n+1\right)7^{n+2}}$$

Remember that to compute a_{n+1} all we do is replace all the n's in a_n with $n+1$. Using the ratio test then gives,

$$L = \left|x-5\right|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$
$$= \left|x-5\right|\lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{(n+1)\,7^{n+2}}\;\frac{n\,7^{n+1}}{(-3)^n}\right|$$
$$= \left|x-5\right|\lim_{n\to\infty}\left|\frac{-3}{(n+1)\,7}\;\frac{n}{1}\right|$$
$$= \frac{3}{7}\left|x-5\right|$$

Now we know that the series will converge if,

$$\frac{3}{7}\left|x-5\right| < 1 \qquad \Rightarrow \qquad \left|x-5\right| < \frac{7}{3}$$

and the series will diverge if,

$$\frac{3}{7}\left|x-5\right| > 1 \qquad \Rightarrow \qquad \left|x-5\right| > \frac{7}{3}$$

In other words, the radius of the convergence for this series is,

$$\rho = \frac{7}{3}$$

As this last example has shown, the radius of convergence is found almost immediately upon using the ratio test.

So, why are we worried about the convergence of power series? Well in order for a series solution to a differential equation to exist at a particular x it will need to be convergent at that x. If it's not convergent at a given x then the series solution won't exist at that x. So, the convergence of power series is fairly important.

Next we need to do a quick review of some of the basics of manipulating series. We'll start with addition and subtraction.

There really isn't a whole lot to addition and subtraction. All that we need to worry about is that the two series start at the same place and both have the same exponent of the x-x_0. If then do then we can perform addition and/or subtraction as follows,

$$\sum_{n=n_0}^{\infty} a_n\left(x-x_0\right)^n \pm \sum_{n=n_0}^{\infty} b_n\left(x-x_0\right)^n = \sum_{n=n_0}^{\infty}\left(a_n \pm b_n\right)\left(x-x_0\right)^n$$

In other words all we do is add or subtract the coefficients and we get the new series.

One of the rules that we're going to have when we get around to finding series solutions to differential equations is that the only x that we want in a series is the x that sits in $\left(x-x_0\right)^n$. This means that we will need to be able to deal with series of the form,

$$\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$$

where c is some constant. These are actually quite easy to deal with.

$$\begin{aligned}\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n &= \left(x-x_0\right)^c \left(a_0 + a_1\left(x-x_0\right) + a_1\left(x-x_0\right)^2 + \cdots\right) \\ &= a_0\left(x-x_0\right)^c + a_1\left(x-x_0\right)^c + a_1\left(x-x_0\right)^{2+c} + \cdots \\ &\sum_{n=0}^{\infty} a_n \left(x-x_0\right)^{n+c}\end{aligned}$$

So, all we need to do is to multiply the term in front into the series and add exponents. Also note that in order to do this both the coefficient in front of the series and the term inside the series must be in the form $x\text{-}x_0$. If they are not the same we can't do this, we will eventually see how to deal with terms that aren't in this form.

Next we need to talk about differentiation of a power series. By looking at (2) it should be fairly easy to see how we will differentiate a power series. Since a series is just a giant summation all we need to do is differentiate the individual terms. The derivative of a power series will be,

$$\begin{aligned}f'(x) &= a_1 + 2a_2\left(x-x_0\right) + 3a_3\left(x-x_0\right)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n a_n \left(x-x_0\right)^{n-1} \\ &= \sum_{n=0}^{\infty} n a_n \left(x-x_0\right)^{n-1}\end{aligned}$$

So, all we need to do is just differentiate the term inside the series and we're done. Notice as well that there are in fact two forms of the derivative. Since the $n=0$ term of the derivative is zero it won't change the value of the series and so we can include it or not as we need to. In our work we will usually want the derivative to start at $n=1$, however there will be the occasion problem were it would be more convenient to start it at $n=0$.

Following how we found the first derivative it should make sense that the second derivative is,

$$\begin{aligned}f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=1}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2}\end{aligned}$$

In this case since the $n=0$ and $n=1$ terms are both zero we can start at any of three possible starting points as determined by the problem that we're working.

Next we need to talk about **index shifts**. As we will see eventually we are going to want our power series written in terms of $\left(x-x_0\right)^n$ and they often won't, initially at least, be in that form. To get them into the form we need we will need to perform an index shift.

Index shifts themselves really aren't concerned with the exponent on the *x* term, they instead are concerned with where the series starts as the following example shows.

Example 2 Write the following as a series that starts at *n=0* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

An index shift is a fairly simple manipulation to perform. First we will notice that if we define *i=n-3* then when *n=3* we will have *i=0*. So what we'll do is rewrite the series in terms of *i* instead of *n*. We can do this by noting that *n=i+3*. So, everywhere we see an *n* in the actual series term we will replace it with an *i+3*. Doing this gives,

$$\begin{aligned}\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+3-1}\left(x+4\right)^{i+3+2}\\ &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+2}\left(x+4\right)^{i+5}\end{aligned}$$

The upper limit won't change in this process since infinity minus three is still infinity.

The final step is to realize that the letter we use for the index doesn't matter and so we can just switch back to *n*'s.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=0}^{\infty}\left(n+3\right)^2 a_{n+2}\left(x+4\right)^{n+5}$$

Now, we usually don't go through this process to do an index shift. All we do is notice that we dropped the starting point in the series by 3 and everywhere else we saw an *n* in the series we increased it by 3. In other words, all the *n*'s in the series move in the opposite direction that we moved the starting point.

Example 3 Write the following as a series that starts at *n=5* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

To start the series to start at *n=5* all we need to do is notice that this means we will increase the starting point by 2 and so all the other *n*'s will need to decrease by 2. Doing this for the series in the previous example would give,

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=5}^{\infty}\left(n-2\right)^2 a_{n-3}\left(x+4\right)^{n}$$

Now, as we noted when we started this discussion about index shift the whole point is to get our series into terms of $\left(x-x_0\right)^n$. We can see in the previous example that we did exactly that with an index shift. The original exponent on the *(x+4)* was *n+2*. To get this down to an *n* we needed to decrease the exponent by 2. This can be done with an index that increases the starting point by 2.

Let's take a look at a couple of more examples of this.

Example 4 Write each of the following as a single series in terms of $\left(x-x_0\right)^n$.

(a) $\left(x+2\right)^2 \sum_{n=3}^{\infty} n a_n \left(x+2\right)^{n-4} - \sum_{n=1}^{\infty} n a_n \left(x+2\right)^{n+1}$

(b) $x\sum_{n=0}^{\infty} \left(n-5\right)^2 b_{n+1} \left(x-3\right)^{n+3}$

Solution

(a) First, notice that there are two series here and the instructions clearly ask for only a single series. So, we will need to subtract the two series at some point in time. The vast majority of our work will be to get the two series prepared for the subtraction. This means that the two series can't have any coefficients in front of them (other than one of course…), they will need to start at the same value of *n* and they will need the same exponent on the x-x_0.

We'll almost always want to take care of any coefficients first. So, we have one in front of the first series so let's multiply that into the first series. Doing this gives,

$$\sum_{n=3}^{\infty} n a_n \left(x+2\right)^{n-2} - \sum_{n=1}^{\infty} n a_n \left(x+2\right)^{n+1}$$

Now, the instructions specify that the new series must be in terms of $\left(x-x_0\right)^n$, so that's the next thing that we've got to take care of. We will do this by an index shift on each of the series. The exponent on the first series needs to go up by two so we'll shift the first series down by 2. On the second series will need to shift up by 1 to get the exponent to move down by 1. Performing the index shifts gives us the following,

$$\sum_{n=1}^{\infty} \left(n+2\right) a_{n+2} \left(x+2\right)^{n} - \sum_{n=2}^{\infty} \left(n-1\right) a_{n-1} \left(x+2\right)^{n}$$

Finally, in order to subtract the two series we'll need to get them to start at the same value of *n*. Depending on the series in the problem we can do this in a variety of ways. In this case let's notice that since there is an *n-1* in the second series we can in fact start the second series at *n=1* without changing its value. Also note that in doing so we will get both of the series to start at *n=1* and so we can do the subtraction. Our final answer is then,

$$\sum_{n=1}^{\infty} \left(n+2\right) a_{n+2} \left(x+2\right)^{n} - \sum_{n=1}^{\infty} \left(n-1\right) a_{n-1} \left(x+2\right)^{n} = \sum_{n=1}^{\infty} \left[\left(n+2\right) a_{n+2} - \left(n-1\right) a_{n-1}\right] \left(x+2\right)^{n}$$

(b) In this part the main issue is the fact that we can't just multiply the coefficient into the series this time since the coefficient doesn't have the same form as the term inside the series. Therefore, the first thing that we'll need to do is correct the coefficient so that we can bring it into the series. We do this as follows,

$$\begin{aligned} x\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} &= (x-3+3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \\ &= (x-3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} + 3\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \end{aligned}$$

We can now move the coefficient into the series, but in the process of we managed to pick up a second series. This will happen so get used to it. Moving the coefficients of both series in gives,

$$\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+4} + \sum_{n=0}^{\infty}3(n-5)^2 b_{n+1}(x-3)^{n+3}$$

We now need to get the exponent in both series to be an n. This will mean shifting the first series up by 4 and the second series up by 3. Doing this gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + \sum_{n=3}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

In this case we can't just start the first series at $n=3$ because there is not an $n-3$ sitting in that series to make the $n=3$ term zero. So, we won't be able to do this part as we did in the first part of this example.

What we'll need to do in this part is strip out the $n=3$ from the second series so they will both start at $n=4$. We will then be able to add the two series together. Stripping out the $n=3$ term from the second series gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + 3(-5)^2 b_1(x-3)^3 + \sum_{n=4}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

We can now add the two series together.

$$75b_1(x-3)^3 + \sum_{n=4}^{\infty}\left[(n-9)^2 b_{n-3} + 3(n-8)^2 b_{n-2}\right](x-3)^{n}$$

This is what we're looking for. We won't worry about the extra term sitting in front of the series. When we finally get around to finding series solutions to differential equations we will see how to deal with that term there.

There is one final fact that we need take care of before moving on. Before giving this fact for power series let's notice that the only way for

$$a + bx + cx^2 = 0$$

to be zero for all x is to have $a=b=c=0$.

We've got a similar fact for power series.

Fact

If,

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = 0$$

for all *x* then,

$$a_n = 0, \quad n = 0, 1, 2, \ldots$$

This fact will be key to our work with differential equations so don't forget it.