Laplace Transforms

There's not too much to this section. We're just going to work an example to illustrate how Laplace transforms can be used to solve systems of differential equations.

Example 1 Solve the following system.

$$x_1' = 3x_1 - 3x_2 + 2 \qquad\qquad x_1(0) = 1$$
$$x_2' = -6x_1 - t \qquad\qquad x_2(0) = -1$$

Solution

First notice that the system is not given in matrix form. This is because the system won't be solved in matrix form. Also note that the system is nonhomogeneous.

We start just as we did when we used Laplace transforms to solve single differential equations. We take the transform of both differential equations.

$$sX_1(s) - x_1(0) = 3X_1(s) - 3X_2(s) + \frac{2}{s}$$
$$sX_2(s) - x_2(0) = -6X_1(s) - \frac{1}{s^2}$$

Now plug in the initial condition and simplify things a little.

$$(s-3)X_1(s) + 3X_2(s) = \frac{2}{s} + 1 = \frac{2+s}{s}$$
$$6X_1(s) + sX_2(s) = -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2}$$

Now we need to solve this for one of the transforms. We'll do this by multiplying the top equation by s and the bottom by -3 and then adding. This gives,

$$\left(s^2 - 3s - 18\right)X_1(s) = 2 + s + \frac{3s^2+3}{s^2}$$

Solving for X_1 gives,

$$X_1(s) = \frac{s^3 + 5s^2 + 3}{s^2(s+3)(s-6)}$$

Partial fractioning gives,

$$X_1(s) = \frac{1}{108}\left(\frac{133}{s-6} - \frac{28}{s+3} + \frac{3}{s} - \frac{18}{s^2}\right)$$

Taking the inverse transform gives us the first solution,

$$x_1(t) = \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right)$$

Now to find the second solution we could go back up and eliminate X_1 to find the transform for X_2 and sometimes we would need to do that. However, in this case notice that the second differential equation is,

$$x_2' = -6x_1 - t \qquad \Rightarrow \qquad x_2 = \int -6x_1 - t\,dt$$

So, plugging the first solution in and integrating gives,

$$\begin{aligned} x_2(t) &= -\frac{1}{18}\int 133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3\,dt \\ &= -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t\right) + c \end{aligned}$$

Now, reapplying the second initial condition to get the constant of integration gives

$$-1 = -\frac{1}{108}\left(133 + 56\right) + c \qquad \Rightarrow \qquad c = \frac{3}{4}$$

The second solution is then,

$$x_2(t) = -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right)$$

So, putting all this together gives the solution to the system as,

$$\begin{aligned} x_1(t) &= \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right) \\ x_2(t) &= -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right) \end{aligned}$$

Compared to the last section the work here wasn't too bad. That won't always be the case of course, but you can see that using Laplace transforms to solve systems isn't too bad in at least some cases.