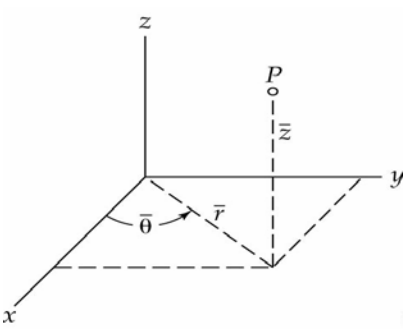
Cylindrical and Spherical Coordinates

Cylindrical Coordinates

cylindrical coordinates (r,θ,z)

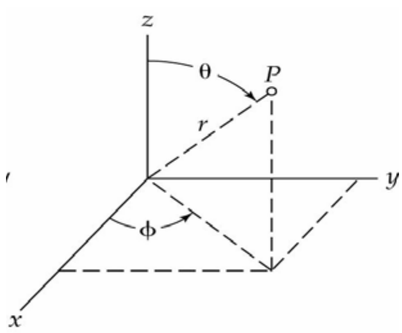
$x = r\cos\theta \quad y = r\sin\theta \quad z = z$



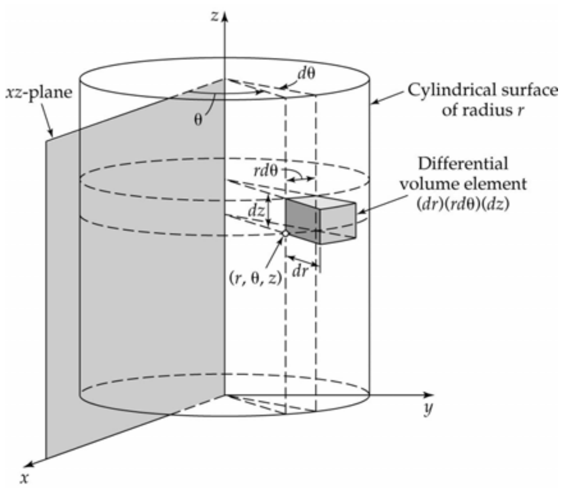
Spherical Coordinates

spherical coordinates (ρ,θ,ϕ)

$x = \rho\cos\phi\sin\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\theta$



Differential Volume in Cylindrical Coordinates



$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_a^b \int_c^d f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Example

Let $E = \{(x,y,z) \mid x^2 + y^2 \le 4, 0 \le z \le 1\}$

Evaluate the triple integral

$$\iiint_E \left(x^2y^2 + 2y^4z\right) dV$$

By using the cylindrical coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

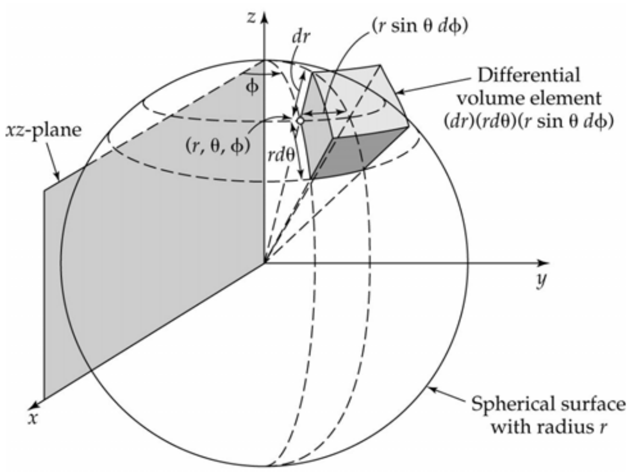
The triple integral becomes:

$$\iiint_E \left(x^2y^2 + 2y^4z\right) dV = \int_0^{2\pi}\int_0^2\left[\int_0^1 \left(x^2y^2 + 2y^4z\right)dz\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left[x^2y^2z + y^4z^2\right]_0^1 r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^2\left[x^2y^2 + y^4\right] r\,dr\right]d\theta = \int_0^{2\pi}\left[\int_0^2 y^2\left(x^2 + y^2\right) r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\int_0^2 r^2\sin^2\theta\left(r^2\right) r\,dr\right]d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^2 r^5\,dr = \frac{32}{3}\pi$$

Differential Volume in Spherical Coordinates



$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Example

Let $E = \left\{ (x,y,z) \,|\, x^2 + y^2 + z^2 \le 4 \right\}$

Evaluate the triple integral

$$\iiint_E \sqrt{x^2 + y^2}\, \exp\left[-\left(x^2 + y^2 + z^2 \right) \right] dV$$

By using the spherical coordinates

$$\begin{cases} x = \rho \sin\theta \cos\phi \\ y = \rho \sin\theta \sin\phi \\ z = \rho \cos\theta \end{cases}$$

The triple integral becomes:

$$\iiint_E \sqrt{x^2 + y^2} \exp\left(- \sqrt{x^2 + y^2 + z^2}\right) dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho \sin\theta \exp(-\rho) \left(\rho^2 \sin\theta \, d\rho \, d\theta \, d\phi\right)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta \, d\theta \int_0^2 \rho^3 \exp(-\rho) \, d\rho = 2\pi^2 \left(3 - 19e^{-2}\right)$$