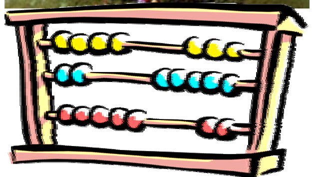
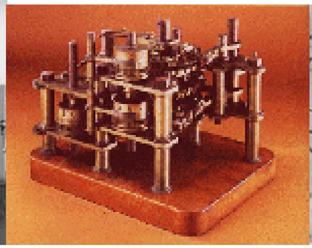
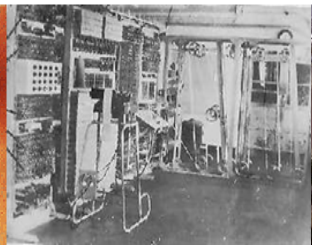
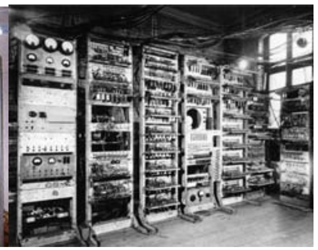




Applied Mathematics Part 02

Sompop Sanongraj, Ph.D

Σομποπ Σανονγραφ, Πη.Δ




Applied Mathematics

Solution Techniques for Ordinary Differential Equations (ODEs)

Sompop Sanongraj

First Order Equations

The I-factor equation: integrating factor method

$$\frac{dy}{dx} + \alpha(x)y = f(x)$$

If there exists an elementary, separable solution, we can rewrite:

$$\frac{d[I(x)y]}{dx} = I(x)f(x)$$

The solution can be obtained by direct integration:

$$y = \frac{1}{I(x)} \int I(x) f(x) dx + \frac{C}{I(x)}$$

where C is an arbitrary constant of integration.

To prove this solution exists, we need to specify I(x).

I(x) = ?

From

$$\frac{d\left[I(x)y\right]}{dx} = I(x)f(x)$$

We get,

$$\frac{dy}{dx} + \frac{1}{I(x)}\frac{dI(x)}{x}y = f(x)$$

Therefore,

$$\frac{1}{I(x)}\frac{dI(x)}{dx} = \alpha(x)$$

Then,

$$I(x) = \exp(\int \alpha(x)dx)$$

where I(X) is called the integrating factor.

Exact Solutions

Occasionally, a solution exists which is an exact differential

$$d\varphi(x, y) = 0 \qquad \textbf{Eq. 0}$$

According to the chain rule,

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = 0$$

How do we use this information to find y as a function x?

Using the property of continuous functions, we specify

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)$$

Therefore, suppose there exists an equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

Then

$$M(x, y) = \frac{\partial \varphi}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial \varphi}{\partial y}$$

If φ is to exist as a possible solution, then the necessary and sufficient condition for φ to exist is

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example

Solve the equation

$$\left(2xy^2 + 2\right)dx + \left(2x^2 y + 4y\right)dy = 0$$

$$\left(2xy^2 + 2\right)dx + \left(2x^2y + 4y\right)dy = 0$$

Check :

$$\frac{\partial M}{\partial y} = 4xy, \frac{\partial N}{\partial x} = 4xy$$

Then this eqution is exact.

The unknow function $\varphi(x, y)$ can be described by :

$$\frac{\partial \varphi}{\partial x} = 2xy^2 + 2$$ **Eq. 1**

$$\frac{\partial \varphi}{\partial y} = 2x^2y + 4y$$ **Eq. 2**

First, we integrate Eq. 1with respect to x (holding y constant)

$$\varphi = x^2y^2 + 2x + f(y) \qquad \text{Eq. 3}$$

Next, we insert Eq. 3 into Eq. 2

$$2x^2y + \frac{df(y)}{dy} = 2x^2y + 4y$$

$$f(y) = 2y^2 + C_2 \qquad \text{Eq. 4}$$

Finally, adding Eq. 4 into Eq. 3 yields

$$\varphi = x^2y^2 + 2x + 2y^2 + C_2$$

Since Eq. 0 integrates to yield φ = C_1, then φ also equals to some arbitary constant. Combining C1 and C2 into another arbitrary constant yields

$$x^2y^2 + 2x + 2y^2 = K = C_1 - C_2$$

$$y^2 = \frac{(K - 2x)}{(x^2 + 2)}$$

$$y = \pm\sqrt{\frac{(K - 2x)}{(x^2 + 2)}}$$

Equations Composed of Homogeneous Functions

The first order equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is said to be homogeneous if P and Q are both homogeneous of the same degree n, for some constant n (including zero).

This implies that first order equations composed of homogeneous functions can always be arranged in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Example

The nonlinear equation

$$y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

can be rearranged to the form

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x} - 1\right)}$$

which is clearly homogeneous. Replacing $y/x = v$ to get,

$$x \frac{dv}{dx} + v = \frac{v^2}{v-1} \quad \text{or} \quad x \frac{dv}{dx} = \frac{v}{v-1}$$

Separation of variables yields

$$\frac{(v-1)}{v}dv = \frac{dx}{x}$$

Integrating term by term produces

$$v - \ln(v) = \ln(x) + \ln(K)$$

where $\ln(K)$ is an arbitrary constant of integration.

$$Kx = \frac{\exp(v)}{v} = \frac{\exp\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)}$$

Bernoulli's Equation

The Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n; \quad n \neq 1$$

is similar to the first order I-factor equation, expect for the nonliear term on the right-hand side, y^n. If we divide y^n throughout, we can obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$$

For the first term, we can show that

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{d(y^{1-n})}{dx}$$

Replacing $y^{1\text{-}n} = v$,

then the original equation is now linear in v

$$\left(\frac{1}{1\text{-}n}\right)\frac{dv}{dx} + P(x)v = Q(x)$$

which is easily solved using the I - factor method

Riccati's Equation

The Riccati's equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

A nonlinear equation which arises in both continuous and staged processes.

A frequently occurring special form is the case when P(x) = -1, then we get

$$\frac{dy}{dx} + y^2 = Q(x)y + R(x) \qquad \textbf{Eq. 5}$$

A change of variables given by

$$y = \frac{1}{u}\frac{du}{dx}$$

Yields the derivative

$$\frac{dy}{dx} = \frac{1}{u}\frac{d^2u}{dx^2} - \frac{1}{u^2}\left(\frac{du}{dx}\right)^2$$

Inserting these into Eq. 5 eliminates the nonlinear term

$$\frac{d^2u}{dx^2} - Q(x)\frac{du}{dx} - R(x)u = 0$$

which is a linear second order equation with nonconstant coefficients. This may be solved and discussed later.

Example

A constant-volume batch reactor undergoes the series reaction sequence

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

The initial concentration of A is denoted by C_{AO}, whereas B and C are initially nil. The reaction rates per unit reactor volume are described by

$$R_A = k_1 C_A^n, \qquad R_B = k_1 C_A^n - k_2 C_B^m$$

Find the solutions of the differential equations describing $C_B(t)$ for the following cases:

(a) $n = 1, m = 2$

(b) $n = 2, m = 1$

(c) $n = 1, m = 1$

Case (a) n =1, m=2

The material balances are written as

$$\frac{dC_A}{dt} = -k_1 C_A$$

$$\frac{dC_B}{dt} = k_1 C_A - k_2 C_B^2$$

The solution for C_A is straightforward

$$C_A = C_{AO} \exp(-k_1 t)$$

Hence, the expression for C_B is nonlinear

$$\frac{dC_B}{dt} = k_1 C_{AO} \exp(-k_1 t) - k_2 C_B^2$$

If we scale time by replacing $\theta = k_2 t$, the above expression becomes identical to the special form of the Riccati equation

$$\frac{dC_B}{d\theta} + C_B^2 = R(\theta)$$

where $Q(\theta) = 0$ and $R(\theta) = \frac{k_1}{k_2} C_{AO} \exp\left(-\frac{k_1}{k_2}\theta\right)$

If we make the Riccati transformation

$$C_B = \frac{1}{u}\frac{du}{d\theta}$$

we finally obtain

$$\frac{d^2u(\theta)}{d\theta^2} - \frac{k_1}{k_2} C_{AO} \exp\left(-\frac{k_1}{k_2}\theta\right) u(\theta) = 0$$

We have thus transformed a nonlinear first order equation to a solvable, linear second order equation

Case (b) n =2, m=1

The simultaneous equations for this case are

$$\frac{dC_A}{dt} = -k_1 C_A^2 \therefore C_A = \left[\frac{C_{AO}}{1+k_1 C_{AO} t}\right] \triangleleft$$

$$\frac{dC_B}{dt} = k_1 C_A^2 \; - k_2 C_B$$

Inserting C_A yields the classic inhomogeneous (I-factor) equation

$$\frac{dC_B}{dt} + k_2 C_B = k_1 \left[\frac{C_{AO}}{1+k_1 C_{AO} t}\right]^2$$

The integrating factor is $\mathrm{I} = \exp(\mathrm{k}_2 t)$; hence, the solution is

$$C_B = k_1 \exp(-k_2 t) \int \exp(k_2 t) \left[\frac{C_{AO}}{1+k_1 C_{AO} t}\right]^2 dt + C \exp(-k_2 t)$$

where C is the constant of integration. The integral is tabulated in the form

$$\int \frac{\exp(ax)}{x^2} dx$$

so we next substitute

$$\tau = 1 + k_1 C_{AO} t,\ dt = \frac{d\tau}{\left(k_1 C_{AO}\right)},\ a = \frac{k_2}{k_1 C_{AO}}$$

hence, we obtain

$$C_B = C_{AO} \exp(-k_2 t) \exp\left(\frac{-k_2}{k_1 C_{AO}}\right) \int \frac{\exp(a\tau)}{\tau^2} d\tau + C \exp(-k_2 t)$$

Performing the integration yields finally

$$C_B = C \exp(-k_2 t) + C_{AO} \exp\left[-\left(k_2 t + \frac{k_2 t}{k_1 C_{AO}}\right)\right]$$

$$\left[-\frac{\exp(a\tau)}{\tau} + a\left(\ln\tau + \frac{a\tau}{(1)(1!)} + \frac{(a\tau)^2}{(2)(2!)} + \frac{(a\tau)^3}{(3)(3!)} + \ldots\right)\right]$$

Now, since $\tau = 1$ when $t = 0$, the arbitary constant C becomes, since $C_B(0) = 0$,

$$C = C_{AO} \exp\left(-\frac{k_2 t}{k_1 C_{AO}}\right)\left[\exp(a) - a\left(\frac{a}{(1)(1!)} + \frac{(a)^2}{(2)(2!)} + \frac{(a)^3}{(3)(3!)} + \ldots\right)\right]$$

Case (c) n =1, m=1

The linear case is described by

$$\frac{dC_A}{dt} = -k_1 C_A \therefore C_A = C_{AO} \exp(-k_1 t)$$

$$\frac{dC_B}{dt} = k_1 C_{AO} \exp(-k_1 t) - k_2 C_B$$

This also yields the I-factor equation, if the time variation is desired.

Often the relationship between C_A and C_B is desired, so we can use a different approach by dividing the two equations to find

$$\frac{dC_B}{dC_A} = -1 + \frac{k_2}{k_1}\left(\frac{C_B}{C_A}\right)$$

This takes the homogeneous form, let

$$\frac{C_B}{C_A} = V, \qquad \frac{dC_B}{dC_A} = C_A \frac{dV}{dC_A} + V$$

hence,

$$C_A \frac{dV}{dC_A} = -1 + \frac{k_2}{k_1}(V) - V$$

so we obtain

$$\frac{dV}{\left[-1+\left(\frac{k_2}{k_1}-1\right)V\right]} = \frac{dC_A}{C_A}$$

Integrating, noting $V = 0$ when $C_A = C_{AO}$ yields finally

$$\frac{C_B}{C_A} = \frac{k_1}{k_2 - k_1}\left[1-\left(\frac{C_A}{C_{AO}}\right)^{\left(\frac{k_2}{k_1}\right)-1}\right]$$

First Order Equations of Second Degree

A nonlinear equation, which is first order and second degree, is

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + y = x - 1$$

This requires a different approach, as nonlinear systems often do.

The first step is to replace $p = dy/dx$ and solve the remaining quadratic equation for p

$$p = \frac{dy}{dx} = 1 \pm \sqrt{x - y}$$

Suggests replacing $u = x - y$, so that we have

$$\frac{du}{dx} = 1 - \frac{dy}{dx}$$

We now have the separable equation

$$\frac{du}{dx} = \pm\sqrt{u}$$

Integrating yields the general solution

$$2\sqrt{u} = \pm x + c$$

Re placing $u = x - y$ shows finally

$$4y = 4x - (c \pm x)^2$$

Again, we observe that the arbitary constant of integration is implicit, which is quite usual for nonlinear systems.

We reinspect the original equation

$$\frac{\mathrm{dy}}{\mathrm{dx}} = 1 \pm \sqrt{x - y}$$

and observe that a solution y = x also satisfies this equation.

This solution cannot be obtained by specializing the arbitray

c, and is thus called a singular solution (Hildebrand, 1965).

This unusual circumstance can only occur in the solution of

nonlinear equations. The singular soltuion sometimes describes

an "envelope" of the family of solutions, but is not in

general a curve belonging to the family of curves

(since it cannot be obtained by specializing the arbitrary constant c)

Solution Methods for Second Order Nonlinear Equations

Some important nonlinear second order equations:

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+y^{\alpha}=0 \text{ (Lane - Emden equation)}$$

$$\frac{d^2\psi}{dt^2}+\omega^2\sin\psi=0 \text{ (Nonlinear Pendulum equation)}$$

$$\frac{d^2y}{dx^2}+ay+by^3=0 \text{ (Duffing equation)}$$

$$\frac{d^2y}{dx^2}+a(y^2-1)\frac{dy}{dx}+y=0 \text{ (Van der Pol equation)}$$

The two most widely used strategies are as follows.

1. Derivative substitution method: replace p = dy/dx if either y is not explicit or x is not explicit.
2. Homogeneous function method: replace v = y/x if the equation can be put into the homogeneous format

$$x\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, \frac{y}{x}\right)$$

Derivative Substitution Method

Example

The nonlinear Pendulum problem

$$\frac{d^2y}{dx^2} + \omega^2 \sin(y) = 0$$

Make the substitution

$$p = \frac{dy}{dx}$$

Therefore,

$$\frac{dp}{dx} + \omega^2 \sin(y) = 0$$

Using the chain rule,

$$\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$$

So we obtain

$$p\frac{dp}{dy} + \omega^2 \sin(y) = 0$$

Integrating yields

$$p^2 = 2\omega^2 \cos(y) + C_1$$

Two brances are possible on taking square roots

$$p = \frac{dy}{dx} = \pm\sqrt{2\omega^2 \cos(y) + C_1}$$

So finally, the integral equation results

$$\int \frac{dy}{\sqrt{2\omega^2 \cos(y) + C_1}} = \pm x + C_2$$

Example

The Fick's law of diffusion for soluble gas (A) reactant dissolves into the flat interface of a deep body of liquid reagent is shown as follows

$$D\frac{d^2C_A}{dz^2} - k_n C_A^n$$

where D is the diffusivity coefficient, C_A is the concentration of gas A, z is the distance of diffusion from the interface, k is the rate constant and n is the integer number

Make a substitution $p = \frac{dC_A}{dz}$, then

$$\frac{d^2C_A}{dz^2} = \frac{dp}{dz} = \frac{dp}{dC_A}\frac{dC_A}{dz} = \frac{dp}{dC_A}p$$

Therefore,

$$p\frac{dp}{dC_A} - \left(\frac{k_n}{D}\right)C_A^n = 0$$

Integrating yields

$$p^2 = 2\left(\frac{k_n}{D}\right)\left(\frac{C_A^{n+1}}{n+1}\right) + C_1$$

Two brances appear. However we must select the negative root becasuse we expect C_A to diminish as we penetrate deeper into the liquid.

$$\frac{dC_A}{dz} = -\sqrt{2\left(\frac{k_n}{D}\right)\left(\frac{C_A^{n+1}}{n+1}\right) + C_1}$$

Since the reaction is irreversible, eventually all of the species A wil be consumed, so that $C_A \to 0$ as $z \to \infty$. Now, as $C_A \to 0$, we expect the flux $(\frac{dC_A}{dz})$ also to diminish to zero. This suggest that we should take $C_1 = 0$, for an unbounded liquid depth; hence,

$$\frac{dC_A}{\sqrt{C_A^{n+1}}} = -\sqrt{\frac{2k_n}{D(n+1)}}dz$$

and this integral is

$$C_A^{(1-n)/2} = -\left(\frac{1-n}{2}\right)\left(\frac{2k_n}{D(n+1)}\right)^{1/2} z + C_2$$

At the interface $(z = 0)$, we denote the gas solubility (Henry's law) as C_A^*, so C_2 is evaluated as

$$C_2 = \left(C_A^*\right)^{(1-n)/2}$$

Example

Find the solution to the linear equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = x$$

The p-substitution method can also be used to good effect on linear equations with nonconstant coefficients, such as the above.

First, replace p = dy/dx to get

$$\frac{dp}{dx} + 2xp = x$$

This is the familiar I-factor linear equation, so let

$$I = \exp\int 2x\,dx = \exp(x^2)$$

hence, the solution for p is

$$p = \exp(-x^2)\int 2x\exp(+x^2)dx + A\exp(-x^2)$$

Noting that $x\,dx = \frac{1}{2}dx^2$ yields

$$p = \frac{1}{2} + A\exp(-x^2)$$

Integrating again produce

$$y = \frac{1}{2}x + A\int \exp(-x^2)dx + B$$

We could replace the indefinite integral with a definite one, since this would only change the already arbitrary constant

$$y = \frac{1}{2}x + A\int_0^x \exp(-\alpha^2)\,d\alpha + B$$

This integral is similar to a tabulated function called the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-\alpha^2)\,d\alpha$$

Using this, we can now write our final solution in terms of known functions and two arbitrary constant

$$y = \frac{1}{2}x + C\operatorname{erf}(x) + B$$

Example

Solve the nonlinear second order equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - x = 0$$

This nonlinear equation can be put into a familiar from,

again by replacing $p = dy/dx$

$$\frac{dp}{dx} + p^2 - x = 0$$

This is the special form of Ricatti's equation with $Q = 0$, $R = x$,

so let $p = \frac{1}{z}\frac{dz}{dx}$ giving the linear equation

$$\frac{d^2z}{dx^2} - xz = 0$$

This is the well-known Airy equation, which will be discussed later.

Homogeneous Function Method

We attempt to rearrange certain equations into the homogeneous format, which carries the dimensional ratio y/x,

$$x\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, \frac{y}{x}\right)$$

If this can be done, then a possible solution may evolve by replacing $v = y/x$.

Often, a certain class of linear equations also obey the homogeneous property, for example, the Euler equation (or Equidimensional equation),

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + By = 0; \qquad A, B \text{ constant}$$

$$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + By = 0; \qquad A, B \text{ constant}$$

Note that units of x cancel in the first two terms. This linear equation with nonconstant coefficients can be reduced to a constant coefficient linear equation by the simple change of variables

$$x = e^t$$

or

$$t = \ln(x)$$

Changing variables starting with the first derivative

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\frac{1}{x}\right) = \frac{d}{dt}\left(\frac{dy}{dt}e^{-t}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}e^{-t}\right)\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2}e^{-t} - \frac{dy}{dt}e^{-t}\right)\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\frac{1}{x^2}$$

Inserting these into the defining equation causes cancellation of x

$$\frac{d^2y}{dt^2} + (A-1)\frac{dy}{dt} + By = 0$$

Now we can use the method of solving the linear constant coefficient equations.

Example

Consider the nonlinear homogeneous equation

$$x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-\left(\frac{y}{x}\right)^2=0$$

Under conditions when the boundary conditions are:

$$\text{BC1:}\qquad \frac{dy}{dx}=1 \quad \text{at } x=1$$

$$\text{BC2:}\qquad y=0 \quad \text{at } x=1$$

Replace $y/x = v$ so that

$$x\left[x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right]+\left[x\frac{dv}{dx}+v\right]^2-v^2=0$$

hence,

$$x^2\frac{d^2v}{dx^2}+2x\frac{dv}{dx}+x^2\left(\frac{dv}{dx}\right)^2+2xv\frac{dv}{dx}=0$$

This has the Euler - Equidmensional form, so let $x = e^t$

$$\left[\frac{d^2v}{dt^2}-\frac{dv}{dt}\right]+2\frac{dv}{dt}+\left(\frac{dv}{dt}\right)^2+2v\left(\frac{dv}{dt}\right)=0$$

Now, since the independent variable (t) is missing, write $p = dv/dt$

$$\frac{d^2v}{dt^2}=\frac{dp}{dt}=\frac{dp}{dv}\frac{dv}{dt}=p\frac{dp}{dv}$$

so that

$$p\frac{dp}{dv}+p+p^2+2vp=0$$

which can be factored to yield two possible solutions

$$p\left[\frac{dp}{dv}+p+(1+2v)\right]=0$$

This can be satified by p = 0, or

$$\frac{dp}{dv}+p=-(1+2v)$$

This latter result is the I-factor equation, which yields for I = exp(v)

$$p=1\text{-}2v+C\exp(\text{-}v)$$

We pause to evaluate C noting

$$p=\frac{dv}{dt}=x\frac{dv}{dx}=\frac{dy}{dx}-\frac{y}{x}$$

hence, at x = 1, then p = 1 and v = y/x = 0, so C = 0.

Integrating again

$$\frac{dv}{1\text{-}2v}=dt$$

yields

$$\frac{K}{\sqrt{1-2v}} = e^{t}$$

$\text{Replacing } v = y/x \text{ and } x = e^{t}$, and since $y = 0$ at $x = 1$, then

$K = 1$ so that squaring yields

$$y = \frac{\left(x^{2}-1\right)}{2x} = \frac{x}{2} - \frac{1}{2x}$$

The singular solution, p = 0, which is dv/dt = 0, so that y/x = constant is a solution. This solution cannot satisfy the two boundary conditions.

Linear Equations of Higher Order

The most general linear differential equation of n^{th} order can be written:

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

where engineers denote f(x) as the forcing function. From the definition of homogeneous type equations, a condition (e.g., boundary condition) or equation is taken to be homogeneous if it is satisfied by y(x) and is also satisfied by Cy(x), where C is an arbitrary constant. Thus the above equation is called the n^{th} order inhomogeneous equation, because of the appearance of f(x). If f(x) = 0, then the above equation is homogeneous.

First, we deal with the unforced, or homogeneous n^{th} order equation

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

The most general solution to the above equation is called the homogeneous or complementary solution (y_c). Noting that: when the forcing function f(x) is present, it produces an additional solution, which is particular to the specific form taken by f(x). Hence, solutions arising because of the presence of finite f(x) are called particular solutions (y_p).

It is clear in the above homogeneous equation that if all coefficients a_o, …$a_{n-1}(x)$ were zero, then we could solve the final equation by n successive integrations of

$$\frac{d^n y}{dx^n} = 0$$

which produces the expression

$$y = C_1 + C_2 x + C_3 x^2 + ... + C_n x^{n-1}$$

containing n arbitrary constants of integration

As a matter of fact, we found that within any defined interval (say, $0 \leq x \leq L$) wherein the coefficients $a_o(x)$, …$a_{n-1}(x)$ are continuous, then there exists a continuous solution to the homogeneous equation containing exactly n independent, arbitrary constants.

Moreover, because the homogeneous equation is linear, it is easily seen that any combination of individual linearly independent solutions is also a solution. We defined linearly independent to mean: an individual solution cannot be obtained from another solution by multiplying it by any arbitrary constant.

For example, the solution $y_1 = c_1\exp(x)$ is linearly independent of $y_2 = c_2\exp(-x)$, since we cannot multiply the latter by any constant to obtain the former. However, the solution $y_3 = 4x^2$ is not linearly independent of $y_4 = 2x^2$, since it is obvious that y_3 can be obtained by multiplying y_4 by 2.

If we denote P as the linear differential operator

$$P = \frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_1(x)\frac{d}{dx} + a_o(x)$$

then we can abbreviate the lengthy representation of the homogeneous n^{th} order equation

$$P[y(x)] = 0$$

Thus, if n linearly independent solutions (y_1, y_2, …y_n) to the associated homogeneous equation:

$$P[y(x)] = 0$$

can be found, then the sum (theorem of superposition)

$$y = C_1 y_1(x) + C_2 y_2(x) + ... + C_n y_n(x) = \sum_{k=1}^{n} C_k y_k(x)$$

is the general solution to the linear, homogeneous, unforced, n^{th} order equation. When we must also deal with the case f(x) ≠ 0, we shall call the above solution the general, complementary solution and denote it as $y_c(x)$.

Thus, it is now clear that if we could find the integral of

$$P\lfloor y_p \rfloor = f(x)$$

where yp is the particular solution, then the complete solution, by superposition,

$$y = y_p(x) + y_c(x) = y_p(x) + \sum_{k=1}^{n} C_k y_k(x)$$

It should now be clear that we have satisfied the original forced equation, since

$$Py = P(y_p + y_c) = Py_p + Py_c = f(x)$$

since by definition

$$Py_c = 0$$

$$Py_p = f(x)$$

Second Order Unforced Equations: Complementary Solutions

The second order linear equation is of great important and arises frequently in engineering.

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_o(x)y = 0$$

For the case of nonconstant coefficients, we can find the complementary solution (y_c) using the general Frobenius series (one of power series) method that will be discussed later.

For the case of constant coefficients (a_o, a_1 = constant), we can find the complementary solution (y_c) using the following method (this method described below is also directly applicable to n^{th} order linear equations provided all constant coefficients).

Thus, for constant coefficients, we shall assume there exists complementary solutions of the form

$$y_c = A\exp(rx); \qquad A, r = \text{constant}$$

where r represents a characteristic root (or eigenvalue) of the equation and A is the integration constant (arbitrary).

Of course! This is necessary that such a proposed solution satisfies the defining equation, so it must be true that

$$\frac{d^2}{dx^2}\left[A\exp(rx)\right] + a_1\frac{d}{dx}\left[A\exp(rx)\right] + a_o\left[A\exp(rx)\right] = 0$$

Performing the indicated operations yields

$$A\left[r^2 + a_1 r + a_o\right]\exp(rx) = 0$$

We thereby deduce that the root(s) must be satisfied by

$$\left[r^2 + a_1 r + a_o\right] = 0$$

then this characteristic equation sustains two roots, given by

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_o}}{2}$$

Since two possible roots exist, then the theorem of superposition suggests that two linearly independent solutions exist for the y_c

$$y_c = A\exp(r_1 x) + B\exp(r_2 x)$$

Are the soltuions linearly independent?

To answer this, we need to know the nature of the two roots.

Are they real or complex? Are they unequal or equal?

Example

Find the complementary solutions (y_c) for the second order equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$$

The characteristic equation is

$$r^2 + 5r + 4 = 0$$

$$r_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4*4}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

Thus, the solution is

$$y_c = A\exp(-x) + B\exp(-4x)$$

It is clear that the roots are real and distinct, so the two solutions are linearly independent.

Example

Solve the second order equation with boundary conditions

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

where y(0) = 0 and dy(0)/dx = 1.

The characteristic equation is

$$r^2 + 4r + 4 = 0$$

so that

$$r_{1,2} = \frac{-4 \pm \sqrt{(4)^2 - 4*4}}{2} = -2$$

which shows that only one root results (i.e., a double root); hence,

we might conclude that the general solution is

$$y_1 = A_o \exp(-2x)$$

Clearly, a single arbitrary constant cannot satisfy the two stated boundary conditions, so it should be obvious that one solution, along with its arbitrary constant, is missing.

As stipulated earlier,an n^{th} order equation

must yield n arbitrary constants, and n linearly independent solutions.

For a present case, $n = 2$, so that we need to find

an additional linearly independent solution.

To find the second solution, we use the definition

of linear independence to propose a new solution, so that we write

$$y_2 = v(x)\exp(-2x)$$

Now, if v(x) is not a simple constant, then the second solution will be linearly independent of $y_1 = A_o \exp(-2x)$. Thus, we have used the first solution to construct the second one. Inserting y_2 into

the defining equation shows after some algebra

$$\frac{d^2v}{dx^2} = 0$$

so that

$$v = Bx + C$$

hence,

$$y_2 = (Bx + C)\exp(-2x)$$

The arbitrary C can be combined with A_o and call it A,

hence, our two linearly indepent solutions yield the

complementary solution

$$y_c = A\exp(-2x) + Bx\exp(-2x)$$

This analysis is in fact a general result for any second order

equation when equal roots occur; that is,

$$y_c = A\exp(rx) + Bx\exp(rx)$$

since the second solution was generated from $y = v(x)\exp(rx)$,

and it is easy to show in general this always leads to $d^2v/dx^2 = 0$.

Applying the boundary conditions,

$$y_c(0) = 0 = A(1) + B(0)(1)$$

hence $A = 0$. To find B, differentiate

$$\frac{dy_c(0)}{dx} = 1 = B(1) + B(0)$$

therefore, $B = 1$; hence, the complementary solution satisfying the stipulated boundary conditions is

$$y_c = x\exp(-2x)$$

Example

Solve the second order equation

$$\frac{d^2y}{dx^2} + y = 0$$

We immediately see difficulties, since the charateristic equation is

$r^2 + 1 = 0$

so complex roots occur

$r_{1,2} = \pm\sqrt{-1} = \pm i$

This defines the complex variable i, we can write the solution

$y_c = A\exp(+ix) + B\exp(-ix)$

This form is not particularly valuable for computation purposes,

it can be put into more useful form by introducing the Euler formula

$e^{ix} = \cos(x) + i\sin(x)$

which allows representation in terms of well-known, transcendental functions. Thus, the complex function e^{ix} can be represented as the linear sum of a real part plus a complex part. This allows us to write

$$y_c = A\left[\cos(x) + i\sin(x)\right] + B\left[\cos(x) - i\sin(x)\right]$$

or

$$y_c = \left(A + B\right)\left[\cos(x)\right] + \left(A - B\right)\left[i\sin(x)\right]$$

Now, since A and B are certainly arbitrary, hence in general $(A + B)$ is different from (A-B)i, then we can define these groups of constants as new constants, so

$$y_c = D\cos(x) + E\sin(x)$$

which is the computationally acceptable general result.

Example

Solve the second order equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

The characteristic equation is

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{2^2 - 4*2}}{2} = 1 \pm i$$

so the solution are

$$y_c = \exp(x)\left[A\exp(+ix) + B\exp(-ix)\right]$$

Introducing the Euler formula as before shows

$$y_c = \exp(x)\left[C\cos(x) + D\sin(x)\right]$$

Particular Solution Methods for Forced Equations

We consider the case of constant coefficients as follows:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_o y = f(x)$$

where again we note the general solution is comprised of two parts,

$$y = y_c(x) + y_p(x)$$

There are three widely used methods to find $y_p(x)$; the first two are applicable only to the case of constant coefficients

1. **Method of Undetermined Coefficients: this is a rather evolutionary technique, which builds on the functional form taken by f(x).**
2. **Method of Inverse Operators: this method builds on the property that integration as an operation is the inverse of differentiation.**
3. **Method of Variation Parameters: this method is the most general approach and can be applied even when coefficients are nonconstant; it is based on the principles of linear independence and superposition, and exploits these necessary properties to construct a particular integral.**

1. Method of Undetermined Coefficients

This widely used technique is somewhat intuitive, and is also easily implemented. The first step in finding y_p is to produce a collection of functions obtained by differentiating f(x). Each of these generated functions are multiplied by an undetermined coefficient and the sum of these plus the original function are then used as a “trial expression” for y_p. The unknown coefficients are determined by inserting the trial solution into the defining equation. Thus, for a second order equation, two differentiation are needed. However, for an n^{th} order equation, n differentiations are necessary (a serious disadvantage).

Example

Find the complementary and particular solutions for the linear equation

$$\frac{d^2y}{dx^2} - y = x^2$$

and evaluate arbitrary constants using y(0) = 1, dy(0)/dx = 0.

For the complementary solution, the characteristic equation is

$$r^2 - 1 = 0; \quad r_{1,2} = \pm 1$$

so the roots are real and distinct; hence,

$$y_c = A\exp(x) + B\exp(-x)$$

To construct the particular solution, we note that repeated differentiation of $f(x) = x^2$ yields x and , so that we porpose the linear combinations

$$y_p = ax^2 + bx + c$$

The undertermined coefficient (a, b, c) are to be determined by inserting our proposed solution into the left-hand of the defining equation; thus,

$$2a - (ax^2 + bx + c) = x^2 + (0)(x) + (0)1$$

Note, we have written f(x) as a descending series with the last two coefficients of magnitude zero. This will help in deducing the values for a, b, c. We next equate all multipliers of x^2 on left- and right-hand sides, all multipliers of x, and

We next equate all multipliers of x^2 on left - and right - hand sides, all multipliers of x, and all multipliers of unity. These deductive operations produce

$$x^2; -a = 1 \therefore a = -1$$

$$x: -b = 0 \therefore b = 0$$

$$1: 2a - c = 0 \therefore c = 2a = -2$$

We solve for the coefficients in sequence and see that

$$y_p = -x^2 - 2$$

The complete solution is then,

$$y = A\exp(x) + B\exp(-x) - \left(x^2 + 2\right)$$

It is clear that all solutions are linearly independent.

Finally, we apply boundary conditions $y(0) = 1$ and

$dy(0)/dx = 0$ to see

$$1 = A + B - 2$$

$$0 = A + B$$

This shows

$$A = B = \frac{3}{2}$$

Example

Find the linearly independent particular solutions for

$$\frac{d^2y}{dx^2} - y = \exp(x)$$

The complementary solution is the same as the previous example. Repeated differentiation of the exponential function reproduces the exponential function. We are keenly aware that a trial solution y_p = a exp(x) is not linearly independent of one of the complementary solutions. We respond to this difficulty by invoking the definition of linear independence

$$y_p = v(x)\exp(x)$$

Clearly, if v(x) is not a constant, then this particular solution will be linearly independent of the complementary function exp(x). Inserting the proposed $y_p(x)$ in the left-hand side of the defining equation yields

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 1$$

To find v(x), we replace dv/dx = p

$$\frac{dp}{dx} + 2p = 1$$

This is the I-factor equation with solution

$$p = \frac{dv}{dx} = \frac{1}{2} + ce^{-2x}$$

Integrating again shows

$$v = \frac{1}{2}x - \frac{c}{2}e^{-2x} + D$$

This suggests a particular solution

$$y_p = \left(\frac{1}{2}x\right)\exp(x)$$

since the other two terms yield contributions that are not linearly independent (they colud be combined with the complementary parts).

The complete solution is

$$y = A\exp(x) + B\exp(-x) + \frac{1}{2}x\exp(x)$$

and all three solutions are linearly independent. Another way to construct the particular integrals under circumstances when the forcing duplicates one of the the complementary solutions is to write

$$y_p = ax\exp(x)$$

Inserting this into the defining equation shows a = ½ as before. In fact, if an identity is not produced (i.e., a is indeterminate), then the next higher power is used, ax^2 exp(x), and so on, until the coefficient is found.

Example

Find the complementary and particular solutions for

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 6xe^{4x}$$

The characteristic equation is

$$r^2 - 8r + 16 = (r-4)^2$$

Thus, we have repeated roots

$$r_{1,2} = 4$$

As we learned earlier, the second complementary solution is obtained by multiplying the first by x, so that

$$y_c = Ae^{4x} + Bxe^{4x}$$

However, the forcing function has the same form as xe^{x},

so our first trial for the y function is

$$y_p = ax^2e^{4x}$$

which is linearly independent of both parts of the complementary solution.

Differentiating twice yields

$$y_p' = 2axe^{4x} + 4ax^2e^{4x}$$

$$y_p'' = 2ae^{4x} + 8axe^{4x} + 8axe^{4x} + 16ax^2e^{4x}$$

inserting these relations into the defining equation yields

$$\left[2a + 16ax + 16ax^2\right]e^{4x} - \left[16ax + 32ax^2\right]e^{4x} + 16\left[ax^2\right]e^{4x} = 6xe^{4x}$$

Cancelling terms shows the null result

$$2ae^{4x} = 6xe^{4x}$$

hence, a is indeterminate. Next, try the higher power

$$y_p = ax^3e^{4x}$$

$$y_p' = 3ax^2e^{4x} + 4ax^3e^{4x}$$

$$y_p'' = 6axe^{4x} + 12ax^2e^{4x} + 12ax^2e^{4x} + 16ax^3e^{4x}$$

Inserting these yields

$$\left[6ax + 24ax^2 + 16ax^3\right]e^{4x} - \left[24ax^2 + 32ax^3\right]e^{4x}$$

$$+ \left[16ax^3\right]e^{4x} = 6xe^{4x}$$

Cancelling terms, what remains identifies the undetermined coefficient

$$6axe^{4x} = 6xe^{4x}$$

hence, $a = 1$.

The complete solution can now be written

$$y = (A + Bx + x^3)e^{4x}$$

We see some serious disadvantages with this technique, especially the large amount of algebraic manipulation (which produces human errors) required for only moderately complex problems.

2. Method of Inverse Operators

This method builds on the Heaviside differential operator, defined as

$$Dy = \frac{dy}{dx}$$

where D is the elementary operator d/dx. It follows certain algebraic laws, and must always precede a function to be operated upon; thus it is clear that repeated differentiation can be represented by

$$D(Dy) = D^2 y = \frac{d^2 y}{dx^2}$$

$$D(D^2 y) = D^3 y = \frac{d^3 y}{dx^3}$$

$$D^n y = \frac{d^n y}{dx^n}$$

Because the operator D is a linear operator, it can be summed and factored

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = D^2y - 8Dy + 16y = 0$$

The operators can be collected together as a larger operator

$$\left(D^2 - 8D + 16\right)y = 0$$

This also can be factored, again maintaining order of operations

$$(D - 4)^2 y = 0$$

In manipulating the Heaviside operator D, the laws of algebraic operation must be followed. These basic laws are as follows.

(a) The Distributive law

For algebraic quantities A, B, C, this law requires

$$A(B+C)=AB+AC$$

We use this above law when we wrote

$$\left(D^2-8D+16\right)y=D^2y-8Dy+16y$$

The operator D is in general distributive.

(b) The Commutative Law

This law sets rules for the order of operation

$$AB = BA$$

which does not generally apply to the Heaviside operator, since obviously

$$Dy \neq yD$$

However, operators do commute with themselves, since

$$(D+4)(D+2) = (D+2)(D+4)$$

(c) The Associative Law

This law sets rules for sequence of peration

$$A(BC) = (AB)C$$

and does not in general apply to D, since sequence for differentiation must be preserved. However, it is true that

$$D(Dy) = (DD)y$$

but that

$$D(xy) \neq (Dx)y$$

since we know that $D(xy) = (Dx)y + xDy$

So far, we have only two rules which must be remembered to use the D operator. We will lead up to these rules gradually by considering, first, the operation on the most prevalent function, the exponential exp(rx). Since we have seen that all complementary solutions have origins in the exponential function.

Operation on Exponential

It is clear that differentiation of exp(rx) yields

$$D(e^{rx}) = re^{rx}$$

and repeated differentiation gives

$$D^2(e^{rx}) = r^2 e^{rx}$$

$$D^n(e^{rx}) = r^n e^{rx}$$

and a sum of operators, forming a polynomial such as P(D)

$$P(D)(e^{rx}) = P(r)e^{rx}$$

$$\left(D^2 + 5D + 4\right)e^{rx} = \left(r^2 + 5r + 4\right)e^{rx} \quad ;\text{see the characteristic equation}$$

Operation on Products with Exponential

The second building block to make operators useful for finding particular integrals is the operation on a general function f(x).

$$D(f(x)e^{rx}) = e^{rx}Df(x) + f(x)D(e^{rx}) = e^{rx}(D+r)f(x)$$

Repeated differentiation can be shown to yield

$$D^2(f(x)e^{rx}) = e^{rx}(D+r)^2 f(x)$$

$$D^n(f(x)e^{rx}) = e^{rx}(D+r)^n f(x)$$

and for any polynomial of D, say P(D)

$$P(D)(f(x)e^{rx}) = e^{rx}(D+r)f(x)$$

The Inverse Operator

Modern calculus often teaches that integration as an operation is the inverse of differentiation. To see this, write

$$\frac{d}{dx}\int f(x)dx = D\int f(x)dx = f(x)$$

which implies

$$\int f(x)dx = D^{-1}f(x)$$

Thus, the operation $D^{-1}f(x)$ implies integration with respect to x, whereas Df(x) denotes differentiation with respect to x. This "integrator," D^{-1}, can be treated like any other algebraic quantity, provided the rules of algebra, mentioned earlier, are obeyed.

Again, we have already seen that
polynomials of operator D
obey two important rules:

1. Rule 1: $P(D)e^{rx} = P(r)e^{rx}$

2. Rule 2: $P(D)\left(f(x)e^{rx}\right) = e^{rx}P(D+r)f(x)$

* Next, we will show that these rules are also
obeyed by inverse operators.

Example

Find the particular solution for

$$\frac{dy}{dx} - 2y = e^{x}$$

Write this in operator notation

$$(D - 2)y_p = e^{x}$$

hence keeping the order of operation in mind

$$y_p = \frac{1}{D-2} e^{rx}$$

Clearly, any polynomial in the denominator can be expanded into an ascending series by synthetic division; in the present case, we can use the binomial theorem written generally as

$$(1+f)^{p} = 1 + pf + \frac{p(p-1)}{(1)(2)} f^{2} + \frac{p(p-1)(p-2)}{(1)(2)(3)} f^{3} + ... +$$

To put our polynomial operator in this form, write

$$\frac{1}{D-2}=\frac{1}{-2\left(1+\frac{D}{2}\right)}$$

so that we see the equivalence f = D/2, p = -1; hence,

$$\frac{1}{-2\left(1+\frac{D}{2}\right)}=-\frac{1}{2}\left[1+\left(\frac{1}{2}D\right)+\left(\frac{1}{2}D\right)^{2}+\left(\frac{1}{2}D\right)^{3}+\ldots\right]$$

hence operating on exp(x) using Rule 1

$$y_p=\frac{1}{D-2}e^{x}=-\frac{1}{2}\left[1+\left(\frac{1}{2}D\right)+\left(\frac{1}{2}D\right)^{2}+\left(\frac{1}{2}D\right)^{3}+\ldots\right]e^{x}$$

yields

$$y_p=\left(-\frac{1}{2}\right)\left[1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{3}+\ldots\right]e^{x}$$

But the series of terms is a geometrical progression and the sum to infinity is equal to 2, so we have finally

$$y_p = -e^x$$

and the general solution is, since $y_c = A\exp(2x)$

$$y(x) = A\exp(2x) - \exp(x)$$

This example simply illustrates that Rule 1 is also applicable to inverse operators.

Rule 1: Inverse Operators

$$\frac{1}{P(D)} e^{rx} = \frac{1}{P(r)} e^{rx}$$

Thus, we could have applied this rule directly to the example without series expansion; since $r = 1$, we have

$$y_p = (D - 2)^{-1} e^{x} = -e^{x}$$

which is quite easy and efficient to use

Occasionally, when appling Rule 1 to find a particular integral yp, we encounter the circumstance P(r) = 0. This is important fail-safe feature of the inverse operator method, since it tells the anylyst that the requirements of linear independence have failed. The case when P(r) = 0 arises when the forcing function f(x) is of the exact form as one the complementary solutions.

Rule 2: Inverse Operators

If P(r) = 0 then obviously P(D) contains a root equal to r; that is, if we could factor P(D) then

$$\frac{1}{P(D)} = \frac{1}{(D-r)}\frac{1}{g(D)}$$

For n repeated roots, this would be written

$$\frac{1}{P(D)} = \frac{1}{(D-r)^n}\frac{1}{g(D)}$$

Now, since g(D) contains no roots r, then Rule 1 can be used.

However, we must modify operation of $1/(D\text{-}r)^n$ when it operates on exp(rx). Thus, we plan to operate on exp(rx) in precise sequence.

Consider Rule 2 for polynomial in the denominator

$$\frac{1}{P(D)}\left[f(x)e^{rx}\right] = e^{rx}\frac{1}{P(D+r)}f(x)$$

and suppose f(x) = 1, then if P(D) = $(D\text{-}r)^n$, we have

$$\frac{1}{(D\text{-}r)^n}\left[(1)e^{rx}\right] = e^{rx}\frac{1}{D^n}(1)$$

This suggests n repeated integrations of unity

$$\frac{1}{D^n}(1) = \iiint_n \ldots \int 1dx = \frac{x^n}{n!}$$

Now, reconsider the general problem for a forcing function exp(rx)

$$\frac{1}{P(D)}\exp(rx) = \frac{1}{(D-r)^n g(D)}\exp(rx)$$

First, operate on exp(rx) using Rule 1 as $g(D)^{-1}$ exp(rx), then shift exp(rx) to get

$$\frac{1}{(D-r)^n}\exp(rx)\frac{1}{g(r)}$$

Next, operate on exp(rx) using Rule 2, taking $f(x) = 1$; hence (since g(r) is finite),

$$\exp(rx)\frac{1}{D^n}\frac{1}{g(r)} = \frac{\exp(rx)}{g(r)}\iiint_n \ldots \int dx = \frac{e^{rx}}{g(r)}\frac{x^n}{n!}$$

We finally conclude, when roots of the complementary solutions appear as the argument in exponential forcing function, we will arrive at P(r) = 0, implying loss of linear independence. By factoring out such roots, and applying Rule 2, a particular solution can always be obtained.

Example

Find the particular solution for

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = xe^{2x}$$

Applying the operator D and factoring

$$\left(D^2 - 4D + 4\right)y_p = \left(D-2\right)^2 y_p = xe^{2x}$$

and solve for y_p

$$y_p = \frac{1}{\left(D-2\right)^2} xe^{2x}$$

If we apply Rule 1, we see $P(2-2) = 0$. So, apply Rule 2,

noting that $f(x) = x$, hence replacing $(D-2)$ with $(D+2-2)$

$$y_p = e^{2x}\frac{1}{\left(D+2-2\right)^2} x$$

$$y_p = e^{2x}\frac{1}{D^2} x = \frac{x^3 e^{2x}}{6}$$

As we saw earlier, for repeated roots, the general complementary solution is $(A + Bx)\exp(2x)$, so that the complete solution is

$$y = (A + Bx)e^{2x} + \frac{1}{6}x^3e^{2x}$$

We can see clearly that speed and efficiency of this method compared to the tedious treatment required by the method of undetermined coefficients.

3. Method of Variation of Parameters

This method can be applied even when coefficients are nonconstant, so that we treat the general case

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_o(x)y = f(x)$$

First, it is assumed that the two linearly independent complementary solutions are known

$$y_c(x) = Au(x) + Bv(x)$$

The Variation of Parameters method is based on the premise that the particular solutions are linearly independent of u(x) and v(x). We start by proposing

$$y_p(x) = F_u(x)u(x) + F_v(x)v(x)$$

where obviously F_u and F_v are not constant. It is clear that if we insert this proposed solution into the defining equation, we shall obtain one equation, but we have two unknowns: F_u and F_v. Thus, we must propose one additional equation, as we show next, to have a solvable system. Performing the required differentiation shows using prime to denote differentiation

$$\frac{dy_p}{dx} = \left(uF_u' + vF_v'\right) + \left(u'F_u + v'F_v\right)$$

It is clear that a second differentiation will introduce second derivatives of the unknown functions F_u, F_v. To avoid this complication, we take as our second proposed equation

$$uF_u' + vF_v' = 0$$

This is the most convenient choice, as we can verify. We next find y_p''

$$\frac{d^2y_p}{dx^2} = \left(F_u u'' + F_v v''\right) + \left(F_u' u' + F_v' v'\right)$$

Inserting dy_p/dx and d^2y_p/dx^2 into the defining equation we obtain, after rearrangement

$$F_u\left[u''+a_1(x)u'+a_o(x)u\right]+F_v\left[v''+a_1(x)v'+a_o(x)v\right]$$
$$+F_u'u'+F_v'v'=f(x)$$

It is obvious that the bracketed terms vanish, because they satisfy the homogenous equation [when f(x) = 0] since they are complementary solutions. The remaining equation has two unknowns,

$$F_u'u'+F_v'v'=f(x) \qquad \textbf{(Eq. a)}$$

This coupled with our second proposition

$$uF_u'+vF_v'=0$$

forms a system of two equations with two unknown. Solving these by defining $p = F_u'$ and $q = F_v'$ shows;

$$p = -\frac{v}{u} q$$

Inserting this into Eq. a gives

$$u'\left(-\frac{v}{u} q\right) + v'q = f(x)$$

hence

$$q = \frac{dF_v}{dx} = \frac{-uf(x)}{u'v - v'u}$$

and this allows p to be obtained as

$$p = \frac{dF_u}{dx} = \frac{vf(x)}{u'v - v'u}$$

There are now separable, so that within an arbitary constant :

$$F_u(x) = \int \frac{vf(x)}{u'v - v'u} dx$$

$$F_v(x) = \int \frac{-uf(x)}{u'v - v'u} dx$$

These integrations, then produce the particular solutions, worth repeating as

$$y_p = u(x)F_u(x) + v(x)F_v(x)$$

Noting that the denominator (u'v - v'u) represent the negative of the so-called Wronskian determinant

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$$

which is nonzero if u and v are indeed linearly independent.

Example

The second order equation with nonconstant coefficients

$$4x\frac{d^2y}{dx^2}+6\frac{dy}{dx}+y=f(x)$$

has complementary solutions (when f(x) = 0) obtainable by the Frobenius series method.

$$y_c(x)=A\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}+B\frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}$$

Find the particular solution when $f(x) = 1/x^{3/2}$

Here, we take the complementary functions to be

$$u = \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}$$

$$v = \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}$$

We first compute the denominator for the integrals

$$u'v - v'u = \frac{1}{2}\frac{1}{x^{3/2}}\left(\cos^2\left(\sqrt{x}\right) + \sin^2\left(\sqrt{x}\right)\right) = \frac{1}{2}\frac{1}{x^{3/2}}$$

Inserting this into the integrals yields:

$$F_u = \int 2\frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}} x^{3/2} \frac{1}{x^{3/2}} dx = 4\sin\left(\sqrt{x}\right)$$

$$F_v = -\int 2\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}} x^{3/2} \frac{1}{x^{3/2}} dx = 4\cos\left(\sqrt{x}\right)$$

so taht we finally have the particular solution

$$y_p = 4\frac{\sin^2\left(\sqrt{x}\right)}{\sqrt{x}} + 4\frac{\cos^2\left(\sqrt{x}\right)}{\sqrt{x}} = \frac{4}{\sqrt{x}}$$

which is linearly independent of the complementary solutions.

Summary of Particular Solution Methods

1. **Method of Undetermined Coefficients**

This technique has advantages for elementary polynomial forcing functions (e.g., $2x^2+1$, $5x^3+3$, etc.) and it is easy to apply and use. However, it becomes quite tedious to use on trigonometric forcing functions, and it is not fail-safe in the sense that some experience is necessary in constructing the trial function. Also, it does not apply to equations with nonconstant coefficients.

2. **Method of Inverse Operators**

This method is the quickest and safest to use with exponential or trigonometric forcing functions. Its main disadvantage is the necessary amount of new material and a student must learn to apply it effectively. Although it can be used on elementary polynomial form), it is quite tedious to apply for such conditions. Also, it cannot be used on equations with nonconstant coefficients.

3. Method of Variation of Parameters

This procedure is the most general method, since it can be applied to equations with variable coefficients. Although it is fail-safe, it often leads to intractable integrals to find F_v and F_u. It is the method of choice when treating forced problems in transport phenomena, since both cylindrical and spherical coordinate systems always lead to equations with variable coefficients.