Partial Differential Equations (PDEs)

- PDEs describe physical processes which depend on more than one variable, e.g.
  - more than one space dimension x, y, z
  - space and time t
- Ubiquitous in engineering systems
  - electrical flow in transmission lines
  - wavefunctions of quantum mechanical particles
  - combustion and flame propagation
  - vibration of strings, cables, beams and plates
  - motion of fluids: water waves, shock waves in aircraft wake
  - heat flow

The second order linear PDEs

- General 2nd order linear PDE for a function u of two variables x and y has the form

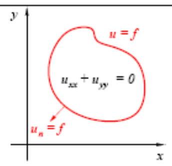
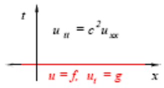
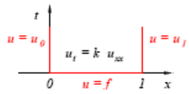
$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g$$

- a, b, c, d, e, f, g can be functions of x and y
- There are three classes of PDE

  $b^2 - 4ac < 0$: **elliptic** – usually smooth solutions, not too hard to solve by computation

  $b^2 - 4ac > 0$: **hyperbolic** – solutions may preserve or develop discontinuities, can be difficult to solve computationally

  $b^2 - 4ac = 0$: **parabolic** – may have features of both

Class	Typical equation	Name	Boundary conditions
Elliptic	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Laplace's equation	<u>Boundary value</u> x, y space coordinates Need one cond. on closed boundary: *Dirichlet*: u given *Neumann*: $\frac{\partial u}{\partial n}$ given  (e.g. steady state temperature distribution)
Hyperbolic	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	Wave equation	<u>Initial value</u> x space, t time Need u and u_t given at $x = 0$ Can also have Neumann or Dirichlet conditions on side boundaries  (e.g. wave on a string)
Parabolic	$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$	Heat equation	<u>Initial/boundary value</u> x space, t time Need u given at $t = 0$ and side conditions (Neumann or Dirichlet)  (e.g. heat conduction in a bar)

How to solve PDEs?

- Most PDEs are too complicated to solve explicitly — nonlinear
- Frequently use numerical methods
  - finite difference methods
  - function approximation methods, e.g. finite element methods
- Two very important considerations affect the solutions of a PDE:
  - Boundary conditions are crucial in governing the solution of PDEs existence of solutions depends on boundary conditions
  - Type of solution depends on class of PDE

How to solve PDEs?

Method of combination of variable:
to reduce the partial differential equation to a single ordinary differential equation (ODE), this can be done only when two of the boundary conditions can be combined into one.

Method of separation of variable:
to reduce the partial differential equation to two ordinary differential equations (ODEs).

Method of separation of variable

Solving the 1D heat equation

- Start off with the 1-space dimensional heat equation, $u = u(x,t)$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- Models the distribution of temperature in a bar of length L with no heat loss from the sides
- Ansatz: assume that the temperature $u(x,t)$ is *separable*

$$u(x,t) = X(x)T(t)$$

- Uniqueness theorem (for linear PDEs): if we find a solution that satisfies the equation and boundary/initial conditions it is the unique solution
- Method works for many different PDEs — very powerful and widely used

Separation of Variables

- First find the partial derivatives of the ansatz

$$\frac{\partial u}{\partial t} = X(x)\frac{\mathrm{d}T(t)}{\mathrm{d}t}$$

$$\frac{\partial u}{\partial x} = \frac{\mathrm{d}X(x)}{\mathrm{d}x}T(t) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}T(t)$$

- Now substitute into the PDE[$\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}, {}' = \frac{\mathrm{d}}{\mathrm{d}x}$]

$$X(x)\dot{T}(t) = kX''(x)T(t)$$

- Rearrange to put all x stuff on one side, all t stuff on the other

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{\dot{T}(t)}{T(t)}$$

Separation of Variables

- lhs is all a function of x, rhs is all a function of t
- x and t are independent: both sides of the equation must be equal to a constant

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\frac{\dot{T}(t)}{T(t)} = -\lambda^2$$

- Separate the variables

$$X''(x) + \lambda^2 X(x) = 0 \qquad \dot{T}(t) + \lambda^2 kT(t) = 0$$

- Now we have two ODEs for the functions X and T, which we can solve!

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$T(t) = C\exp(-\lambda^2 kt)$$