Bessel's Differential Equation

In the **Sturm-Liouville Boundary Value Problem**, there is an important special case called *Bessel's Differential Equation* which arises in numerous problems, especially in polar and cylindrical coordinates. **Bessel's Differential Equation** is defined as:

$$x^2 y'' + xy' + \left(x^2 - n^2\right)y = 0$$

where n is a non-negative real number. The solutions of this equation are called **Bessel Functions** of order n. Although the order n can be any real number, the scope of this section is limited to *non-negative integers*, i.e., $n = 0, 1, 2, 3, \ldots$, unless specified otherwise.

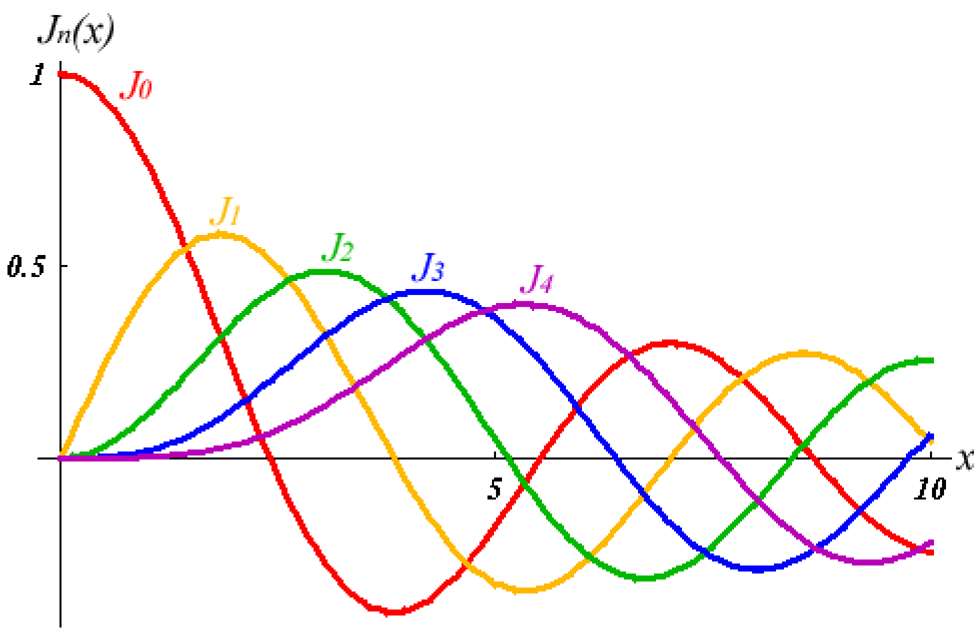
Since Bessel's differential equation is a second order ordinary differential equation, two sets of functions, the Bessel function of the first kind $J_n(x)$ and the Bessel function of the second kind (also known as the Weber Function) $Y_n(x)$, are needed to form the general solution:

$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

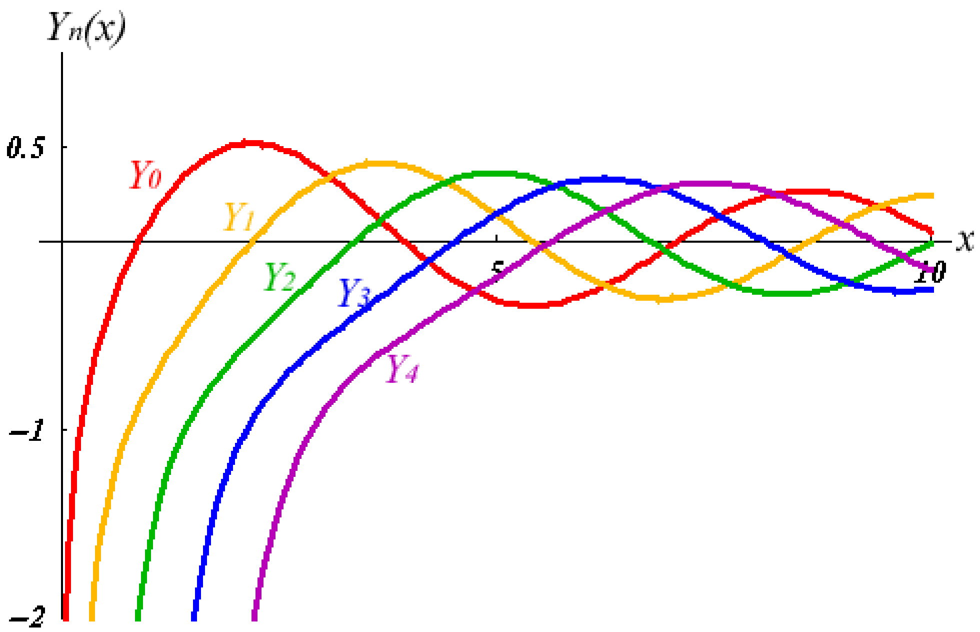
However, $Y_n(x)$ is *divergent* at $x = 0$. The associated coefficient c_2 is forced to be *zero* to obtain a physically meaningful result when there is no source or sink at $x = 0$.

See plots of Bessel Functions

Bessel Functions of the First Kind



Bessel Functions of the Second Kind



Top of Page

Important Properties

Basic Relationship: The Bessel function of the first kind of order n can be expressed as a series of gamma functions.

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right]$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\,\Gamma(n+k+1)}$$

The Bessel function of the second kind of order n can be expressed in terms of the Bessel function of the first kind.

$$Y_n(x)=\frac{2}{\pi}J_n(x)\left(\ln\frac{x}{2}+\gamma\right)-\frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-n}$$

$$+\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m-1}\left[\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\ldots+\frac{1}{m+n}\right)\right]}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n}$$

$$=\lim_{p\to n}\frac{J_p(x)\cos p\pi-J_{-p}(x)}{\sin p\pi}\qquad n=0,1,2,\ldots$$

Recurrence Relation: A Bessel function of higher order can be expressed by Bessel functions of lower orders.

$J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$	$Y_{n+1}(x)=\frac{2n}{x}Y_n(x)-Y_{n-1}(x)$
$J_n'(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$	$Y_n'(x)=\frac{1}{2}\left[Y_{n-1}(x)-Y_{n+1}(x)\right]$
$J_n'(x)=J_{n-1}(x)-\frac{n}{x}J_n(x)$	$Y_n'(x)=Y_{n-1}(x)-\frac{n}{x}Y_n(x)$
$J_n'(x)=\frac{n}{x}J_n(x)-J_{n+1}(x)$	$Y_n'(x)=\frac{n}{x}Y_n(x)-Y_{n+1}(x)$
$\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$	$\frac{d}{dx}\left[x^nY_n(x)\right]=x^nY_{n-1}(x)$
$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$	$\frac{d}{dx}\left[x^{-n}Y_n(x)\right]=-x^{-n}Y_{n+1}(x)$

The Modified Bessel's Differential Equation

Similar to the relations between the trigonometric functions and the hyperbolic trigonometric functions,

$$\sin ix=i\sinh x\qquad\qquad \cos ix=\cosh x\qquad\qquad e^{\pm x}=\cosh x\pm i\sinh x$$

The **modified Bessel's differential equation** is defined in a similar manner by changing the variable x to ix in Bessel's differential equation:

$$x^2y''+xy'-\left(x^2+n^2\right)y=0$$

Its general solution is

$$y(x)=c_1I_n(x)+c_2K_n(x)$$

where

$$I_n(x) = i^{-n} J_n(ix)$$

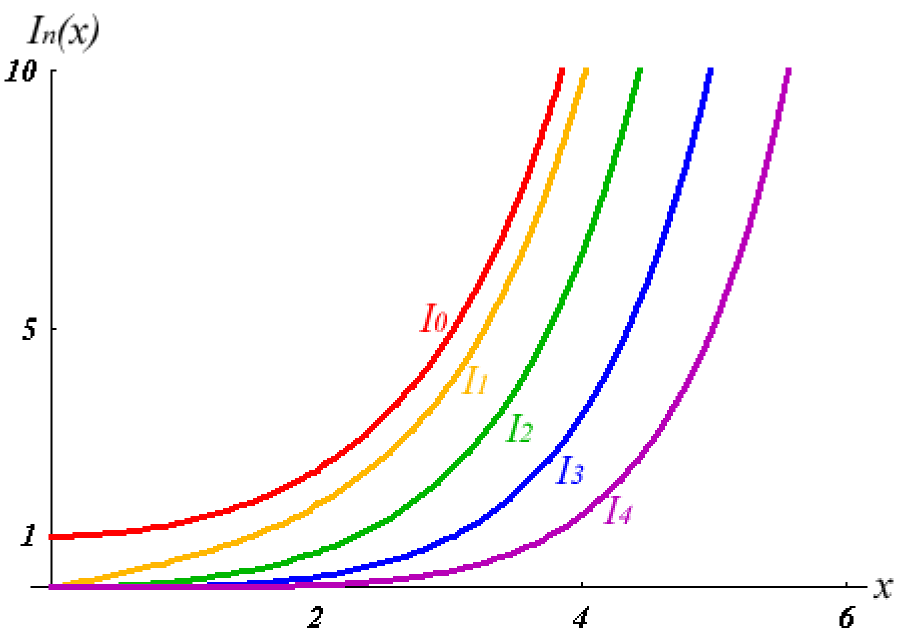
$$K_n(x) = \frac{\pi}{2} i^{n+1} \left[J_n(ix) + i Y_n(ix) \right] = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix)$$

$$= \lim_{p \to n} \frac{\pi}{2} \left[\frac{I_{-p}(x) - I_p(x)}{\sin p\pi} \right]$$

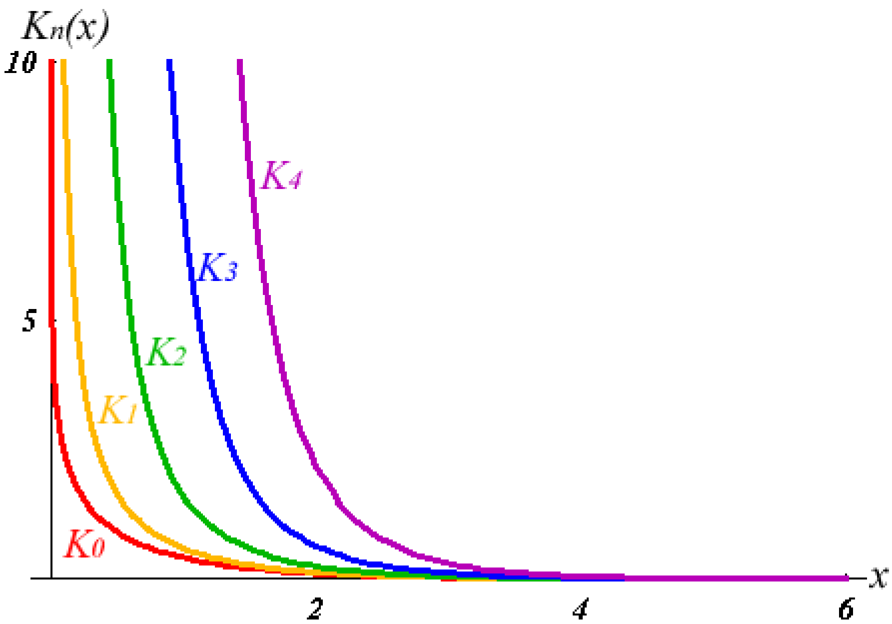
are the **modified Bessel functions** of the first and second kind respectively.

See plots of Modified Bessel Functions

Modified Bessel Functions of the First Kind



Modified Bessel Functions of the Second Kind



Top of Page

Important Properties

Recurrence Relation: A modified Bessel function of higher order can be expressed by modified Bessel functions of lower orders.

$I_{n+1}(x)=I_{n-1}(x)-\frac{2n}{x}I_n(x)$	$K_{n+1}(x)=K_{n-1}(x)+\frac{2n}{x}K_n(x)$
$I_n'(x)=\frac{1}{2}[I_{n-1}(x)+I_{n+1}(x)]$	$K_n'(x)=-\frac{1}{2}[K_{n-1}(x)+K_{n+1}(x)]$
$I_n'(x)=I_{n-1}(x)-\frac{n}{x}I_n(x)$	$K_n'(x)=-K_{n-1}(x)-\frac{n}{x}K_n(x)$
$I_n'(x)=\frac{n}{x}I_n(x)+I_{n+1}(x)$	$K_n'(x)=\frac{n}{x}K_n(x)-K_{n+1}(x)$
$\frac{d}{dx}\left[x^nI_n(x)\right]=x^nI_{n-1}(x)$	$\frac{d}{dx}\left[x^nK_n(x)\right]=-x^nK_{n-1}(x)$
$\frac{d}{dx}\left[x^{-n}I_n(x)\right]=x^{-n}I_{n+1}(x)$	$\frac{d}{dx}\left[x^{-n}K_n(x)\right]=-x^{-n}K_{n+1}(x)$

Top of Page

Top of Page