1304 431 Transport Phenomena

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Example 2: Flow in Circular Pipe

(Set M = Momentum (kg.m/s^2), δ = Thickness of film (m), V_z = Velocity (m/s) and P = Pressure (N/m^2)

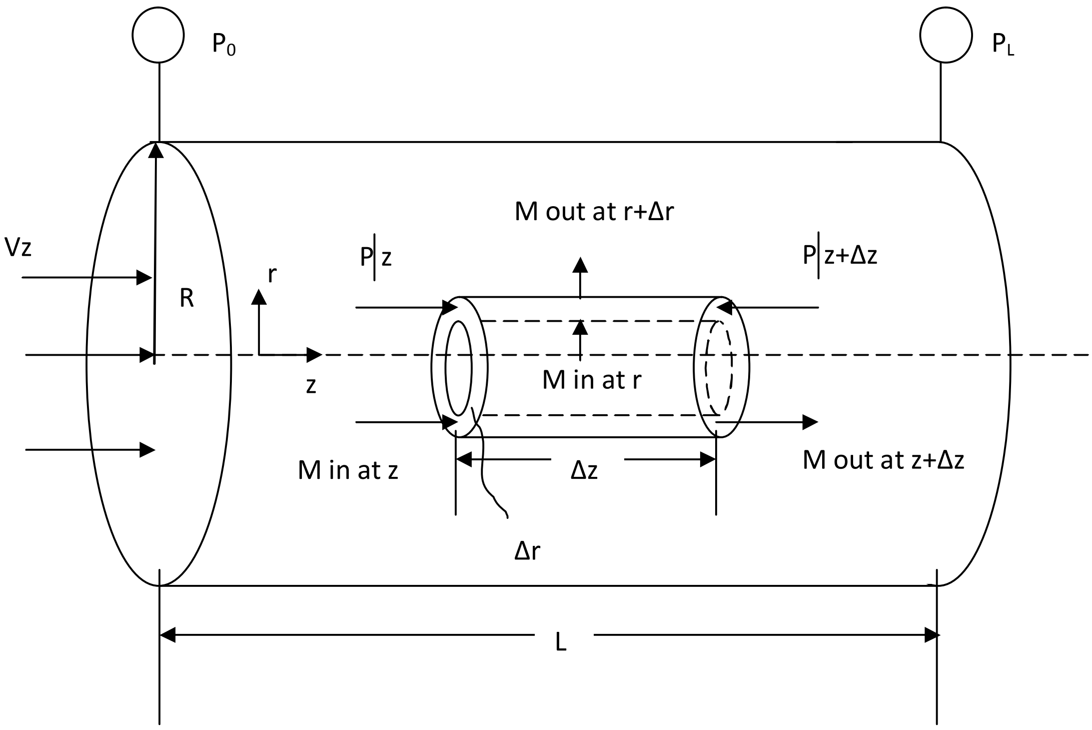


Fig. 2 Flow in circular pipe in horizontal direction

Apply general law of conservation momentum (force balance)

Term 1		Term 2		Term 3
Rate of momentum in Shell in z-direction by convection	-	Rate of momentum out shell in z+Δz by convection	+	Rate of momentum in shell in r-direction by molecular diffusion

	Term 4		Term 5	Term 6
-	Rate of momentum out shell in r+Δr By molecular diffusion	+	Sum of external forces acting on shell	= Rate of momentum accumulation

For term 6 = 0 (assume steady state system)

For term 1: Rate of momentum in shell at z = z by convection

$$(\text{mass flow rate})(v_z)\Big|_{z=z} = (\rho . v_z . 2¶r . \Delta r)v_z\Big|_{z=z}$$

For term 2: Rate of momentum out shell at z= z + Δz by convection

$$(\text{mass flow rate})(v_z)\Big|_{z=z+\Delta z} = (\rho . v_z . 2¶r . \Delta r)v_z\Big|_{z=z+\Delta z}$$

For term 3: Rate of momentum in shell at r = r by diffusion

$$(2¶r . \Delta r)\tau_{rz}\Big|_{r=r}$$

For term 4: Rate of momentum out shell at r = r + Δr

$$(2¶r . \Delta r)\tau_{rz}\Big|_{r=r+\Delta r}$$

For term 5: Sum of external forces acting on shell

$$= \underbrace{(2\P r.\,\Delta r).P\Big|_{z=z} - (2\P r.\,\Delta r).P\Big|_{z=z+\Delta z}}_{\text{Be careful with the sign of each term.}} + \underbrace{\text{gravitational force}}_{\text{Because the flow is in horizontal, no gravitational force}} = 0$$

Be careful with the sign of each term.

Because the flow is in horizontal, no gravitational force

Combine all terms

So, from momentum balance around shell

$$(\rho.v_z.2\pi r.\Delta r)v_z|_z - (\rho.v_z.2\pi r.\Delta r)v_z|_{z+\Delta z} + (2\pi r.\Delta z)\tau_{rz}|_r - (2\pi r.\Delta z)\tau_{rz}|_{r+\Delta r} + (2\pi r.\Delta r)P|_z - (2\pi r.\Delta r)P|_{z+\Delta z} = 0$$

Divided through by 2¶ Δr Δz

$$-\left(\frac{r\rho v_z^2|_{z+\Delta z} - r\rho v_z^2|_z}{\Delta z}\right) - \left(\frac{r\tau_{rz}|_{r+\Delta r} - r\tau_{rz}|_r}{\Delta r}\right) - \left(\frac{rP|_{z+\Delta z} - rP|_z}{\Delta z}\right) = 0$$

Set limit Δr→0, limit Δz→0

$$-\lim_{\Delta z\to 0}\left(\frac{r\rho v_z^2|_{z+\Delta z} - r\rho v_z^2|_z}{\Delta z}\right) - \lim_{\Delta r\to 0}\left(\frac{r\tau_{rz}|_{r+\Delta r} - r\tau_{rz}|_r}{\Delta r}\right) - \lim_{\Delta z\to 0}\left(\frac{rP|_{z+\Delta z} - rP|_z}{\Delta z}\right) = 0$$

$$-\frac{d(\rho r v_z^2)}{dz} - \frac{d(r\tau_{rz})}{dr} - \frac{d(rP)}{dz} = 0$$

For incompressible fluid (ρ = constant)

$$-\frac{\rho d(r v_z^2)}{dz} - \frac{d(r\tau_{rz})}{dr} - \frac{d(rP)}{dz} = 0$$

Since $V_z \neq f(z)$, $\frac{dv_z}{dz} = 0$ and $\frac{d(v_z^2)}{dz} = 0$.

$$-\frac{d(r.\tau_{rz})}{dr} - \frac{d(r.P)}{dz} = 0$$

Simplified equation by approximation $dP \approx \Delta P$ and $dz \approx \Delta z \approx L$ and $r \neq f(z)$:

$$r\frac{dP}{dz} \approx r\frac{\Delta P}{\Delta z} = \frac{r(P_L - P_0)}{L}$$

Then we obtain 1st ODE w.r.t. τ_{rz};

$$-\frac{d(r.\tau_{rz})}{dr} - \frac{r(P_L - P_0)}{L} = 0$$

Apply Newton's Law: $\tau_{rz} = -\frac{\mu dv_z}{dr}$

$$-\frac{d}{dr}\left(r.(-\mu\frac{dv_z}{dr})\right) - \frac{r(P_L - P_0)}{L} = 0$$

For isothermal system, μ=constant

$$\mu\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) + \frac{r(P_0 - P_L)}{L} = 0$$

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{r(P_0 - P_L)}{\mu L}$$

Perform the equation by indefinite integration (1st)

$$\int d\left(r\frac{dv_z}{dr}\right) = \int -\frac{r(P_0 - P_L)}{\mu L}dr$$

$$r\frac{dv_z}{dr} = -\frac{r^2(P_0 - P_L)}{2\mu L} + C_1$$

$$\frac{dv_z}{dr} = -\frac{r(P_0 - P_L)}{2\mu L} + \frac{C_1}{r}$$

Perform the equation by indefinite integration (2nd)

$$\int d(v_z) = \int \frac{-r(P_0 - P_L)}{2\mu L} dr + \int \frac{C_1}{r} dr$$

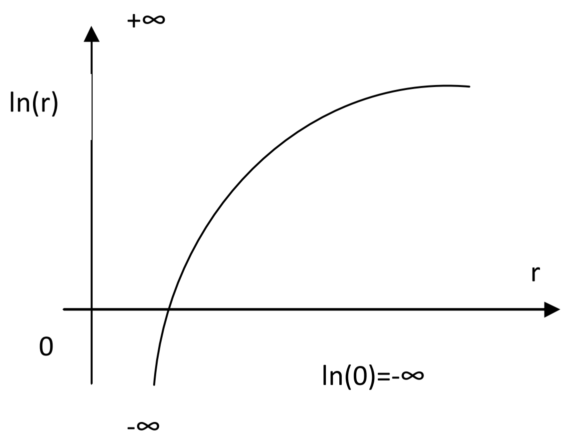
$$v_z = \frac{-r^2(P_0 - P_L)}{4\mu L} + C_1 \ln(r) + C_2$$

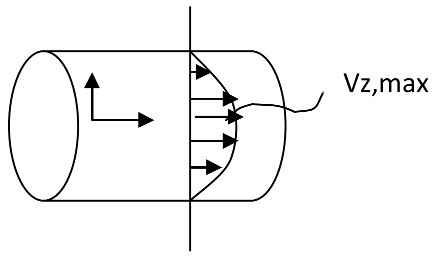
Apply BC1: At r = 0, $V_z = V_{z,\,max}$ (finite)

$$(v_{z,max}) = -\frac{(0)^2(P_0 - P_L)}{4\mu L} + C_1(-\infty) + C_2$$

(finite)

$-\infty$





Note: The above equation is -∞ by means of mathematic, but in fact the equation must be finite because Vz,max is finite. So, we force $C_1 = 0$ in order to valid the equation.

$$v_z = \frac{-r^2(P_0 - P_L)}{4\mu L} + C_1 \ln(r) + C_2$$

Finite = finite + <u>0</u> + C_2

Substitute C_1 into the equation

$$v_z = \frac{-r^2(P_0 - P_L)}{4\mu L} + C_2$$

Apply BC2: At r=R, V_z=0

$$0 = \frac{-R^2(P_0 - P_L)}{4\mu L} + C_2$$

$$C_2 = \frac{R^2(P_0 - P_L)}{4\mu L}$$

Substitute C_2 into the equation

$$v_z = \frac{-r^2(P_0 - P_L)}{4\mu L} + \frac{R^2(P_0 - P_L)}{4\mu L}$$

$$v_z = \frac{(P_0 - P_L)}{4\mu L}(R^2 - r^2)$$

So we get 1st solution as the velocity profile:

$$v_z = \frac{R^2(P_0 - P_L)}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

What is expression of the maximum velocity?

Since BC2 at r=0, Vz = Vz,max, we get,

$$v_{z,max} = \frac{R^2(P_0 - P_L)}{4\mu L}\left[1 - \left(\frac{0}{R}\right)^2\right]$$

$$v_{z,max} = \frac{R^2(P_0 - P_L)}{4\mu L}$$

What is equation of volumetric flow rate (Q)?

$$\int_0^Q dQ = \int_0^R (v_z . 2\pi r . dr)$$

$$Q = \int_0^R \frac{R^2(P_0 - P_L)}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right] . 2\pi r . dr$$

$$Q = \frac{2\pi R^2(P_0 - P_L)}{4\mu L}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]\Bigg|_0^R$$

$$Q = \frac{2\pi R^2(P_0 - P_L)}{4\mu L}\left[\frac{4R^4 - 2R^4}{8R^2}\right]$$

$$Q = \frac{\pi R^4(P_0 - P_L)}{8\mu L} \quad ; \qquad (Q \propto R^4)$$

"The Hagen-Poiseuille equation"

Hydraulic Eng. (1839, German)   Physician (1841, France)

What is mean velocity ($\overline{v_z}$)?

$$\bar{v}_z = \frac{1}{2} v_{z,max}$$

$$\bar{v}_z = \frac{R^2(P_0 - P_L)}{8\mu L}$$

What is Force acting on surface of pipe?

$$\vec{F} = (\tau_{rz}|_{r=R})(2\pi RL)$$

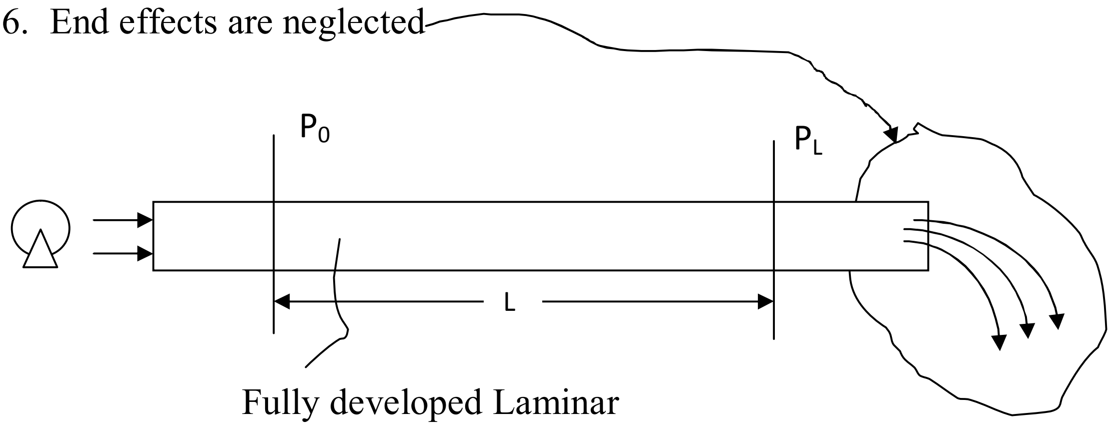
$$\vec{F} = \left(-\mu . -\frac{dv_z}{dr}\right)(2\pi RL)$$

$$\vec{F} = \mu . \frac{R(P_0 - P_L)}{2\mu L}(2\pi RL)$$

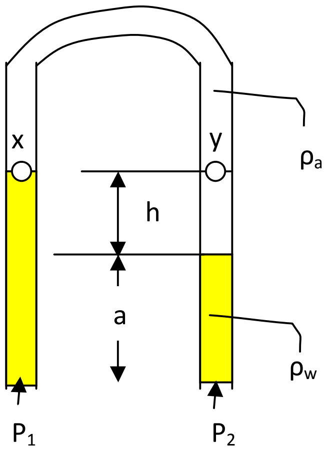
$$\vec{F} = \pi R^2(P_0 - P_L)$$

Assumptions of the Hagen-Poiseuille Equation

1. Isothermal system
2. Laminar flow
3. Incompressible fluid (v, ρ constant)
4. Newtonian fluid
5. Steady state system
6. End effects are neglected



Inverted Monometer



$$P_{high} = P_1$$

$$P_{low} = P_2$$

$$P_1 = \rho_w gh + \rho_w ga + P_x$$

$$P_2 = \rho_a gh + \rho_w ga + P_y$$

$$P_1 - P_2 = \rho_w gh + \rho_w ga + P_x - \rho_a gh - \rho_w ga - P_y$$

Because $x = y$, $P_x = P_y$

$$P_1 - P_2 = (\rho_w - \rho_a)gh$$