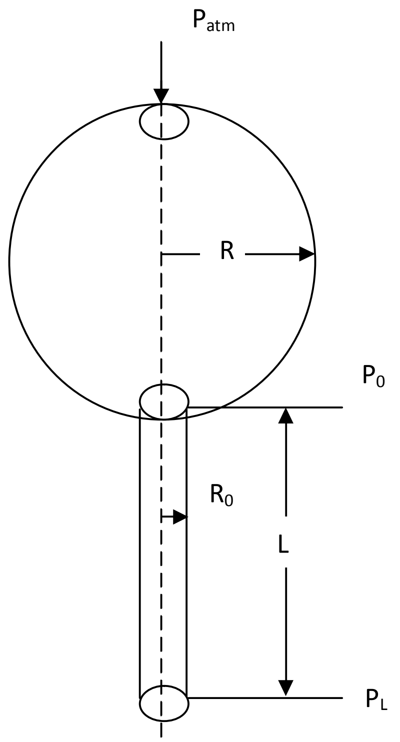
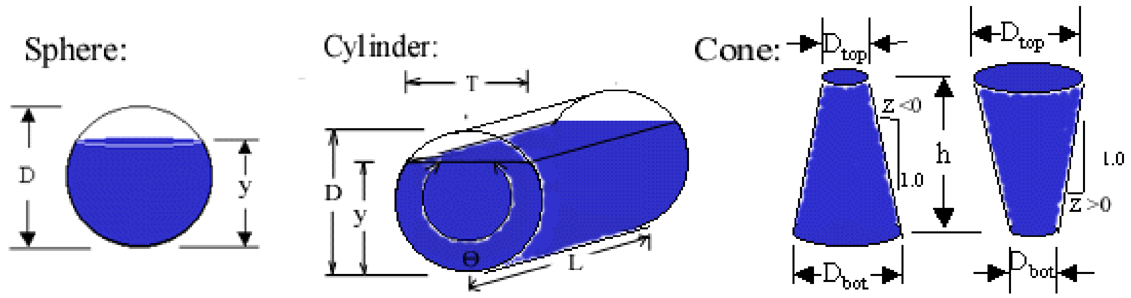
Example 7 Drain water from spherical tank





Sphere : $V = \frac{\pi}{3} y^2 (1.5D - y)$

Cylinder : $V = \frac{L D^2}{8} (\theta - \sin(\theta))$ $\quad T = 2\sqrt{y(D-y)} = D \sin\left(\frac{\theta}{2}\right)$

Cone : $V = \frac{\pi h}{12} (D_{bot}^2 + D_{bot} D_{top} + D_{top}^2)$ $\quad z = \frac{1}{2h} (D_{top} - D_{bot})$

Given: mass of liquid inside the spherical tank at anytime "t"

$$m = \pi R h^2 \left(1 - \frac{1}{3}\frac{h}{R}\right)\rho$$

Mass balance around tank

(in) – (out) + (produced) = (accumulation)

$$0 - \rho Q + 0 = \frac{d(m)}{dt}$$

$$-\rho Q = \frac{d}{dt}\pi R h^2 \left(1 - \frac{1}{3}\frac{h}{R}\right)\rho$$

Assume incompressible fluid, ρ is constant.

$$-Q = \frac{d}{dt}\pi R h^2 \left(1 - \frac{1}{3}\frac{h}{R}\right)$$

$$-Q = \pi R \frac{d}{dt} h^2 \left(1 - \frac{1}{3}\frac{h}{R}\right)$$

$$-Q = \pi R \frac{d}{dt}\left(h^2 - \frac{1}{3}\frac{h^3}{R}\right)$$

$$-Q = \pi R \frac{d}{dt}(f(h)) \qquad (1)$$

By chain rule,

$$\frac{d}{dt}f(h) = \frac{df(h)}{dh}\frac{dh}{dt}$$

Equation 1,

$$-Q = \left(\frac{d}{dt}\left(h^2 - \frac{1}{3}\frac{h^3}{R}\right)\frac{dh}{dt}\right)\pi R$$

$$-Q = \left(\left(2h - \frac{h^2}{R}\right)\frac{dh}{dt}\right)\pi R$$

Momentum balance around shell in pipe (From example 3)

$$Q = \frac{\pi R_0^4 \rho g (h+L)}{8\mu L} \qquad (2)$$

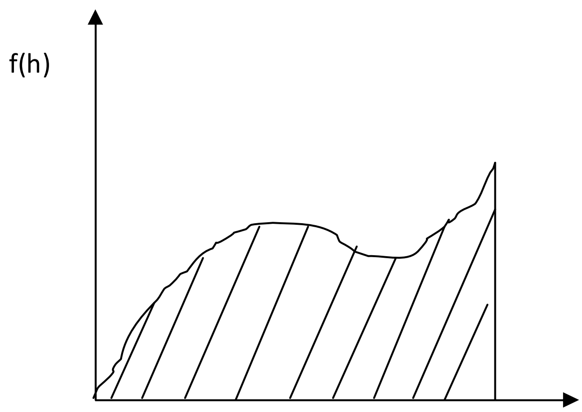
Set Equation (2) = Equation (1) ,

$$\frac{\pi R_0^4 \rho g (h+L)}{8\mu L} = \left(-\left(2h - \frac{h^2}{R}\right)\frac{dh}{dt}\right)\pi R$$

$$-\int_0^{t_f} \frac{R_0^4 \rho g}{8\mu L} dt = \int_{h=2R}^{0} R \frac{\left(2h - \frac{h^2}{R}\right)}{(h+L)} dh$$

$$t_f = \frac{\int_{h=2R}^{0} \frac{\left(2h - \frac{h^2}{R}\right) dh}{(h+L)}}{-\frac{R_0^4 \rho g}{8\mu L}}$$

(1) Graphical Method (In case that we know the value of every variables)



R

0 2R

(2) Analytical Method: set h+L = u, du = dh

$$\int_{h=2R}^{h=o} \frac{R\left(2h-\frac{h^2}{R}\right)}{h+L}dh = \int \frac{Rh(2R-h)}{R(h+L)}dh$$

$$= \int_{u_1=2R+L}^{u_2=L} \frac{(u-L)(2R-u+L)du}{(u-L+L)}$$

$$= \int_{u_1=2R+L}^{u_2=L} \left(\frac{2Ru-u^2+uL-2RL+uL-L^2}{u}\right)du$$

$$= \int_{u_1}^{u_2} \left(2R-u+2L-\frac{2RL}{u}-\frac{L^2}{u}\right)du$$

$$= 2Ru-\frac{u^2}{2}+2Lu-2RL\ln(u)-L^2\ln u\Big|_{u_1}^{u_2}$$

$$= 2R(h+L)-\frac{(h+L)^2}{2}+2Lu-2RL\ln(u)-L^2\ln u|_{h=2R}^{h=0}$$

$$= 2R(h+L)-\frac{(h+L)^2}{2}+2L(h+L)-2RL\ln(h+L)-L^2\ln(h+L)|_{h=2R}^{h=0}$$

$$= 2Rh-2RL-\frac{h^2}{2}-\frac{2hL}{2}-\frac{L^2}{2}+2Lh+2L^2-2RL\ln(h+L)-L^2\ln(h+L)|_{2R}^{0}$$

$$= \left[0 - 2RL - 0 - 0 - \frac{L^2}{2} + 0 - 2RL\ln(L) - L^2\ln(L)\right]$$

$$- \left[4R^2 - 2RL - \frac{4R^2}{2} - \frac{4RL}{2} - \frac{L^2}{2} + 4RL + 2L^2\right.$$

$$\left. - 2RL\ln(2R + L) - L^2\ln(2R + L)\right]$$

$$= L^2\left[\frac{-2R}{L}\left(1 + \frac{R}{L}\right) + \left(1 + \frac{2R}{L}\right)\ln\left(1 + \frac{2R}{L}\right)\right]$$

$$t_f = \frac{L^2}{\rho g R^4/(8\mu L)}\left[\frac{2R}{L}\left(1 + \frac{R}{L}\right) - \left(1 + \frac{2R}{L}\right)\ln\left(1 + \frac{2R}{L}\right)\right]$$