Energy Transport

A statement of the law of conservation of energy (for steady-state):

$$
\begin{pmatrix}\text{Rate of energy in}\\ \text{by convective}\\ \text{transport}\end{pmatrix}
-
\begin{pmatrix}\text{Rate of energy out}\\ \text{by convective}\\ \text{transport}\end{pmatrix}
+
\begin{pmatrix}\text{Rate of energy in}\\ \text{by molecular}\\ \text{transport}\end{pmatrix}
-
\begin{pmatrix}\text{Rate of energy out}\\ \text{by molecular}\\ \text{transport}\end{pmatrix}
$$

$$
\begin{pmatrix}\text{Rate of work}\\ \text{done on system}\\ \text{by molecular}\\ \text{transport}\end{pmatrix}
-
\begin{pmatrix}\text{Rate of work}\\ \text{done on system}\\ \text{by molecular}\\ \text{transport}\end{pmatrix}
+
\begin{pmatrix}\text{Rate of work}\\ \text{done on system}\\ \text{by external}\\ \text{forces}\end{pmatrix}
+
\begin{pmatrix}\text{rate of energy}\\ \text{production}\end{pmatrix}
= 0
$$

Example 8 Plane wall with heat source

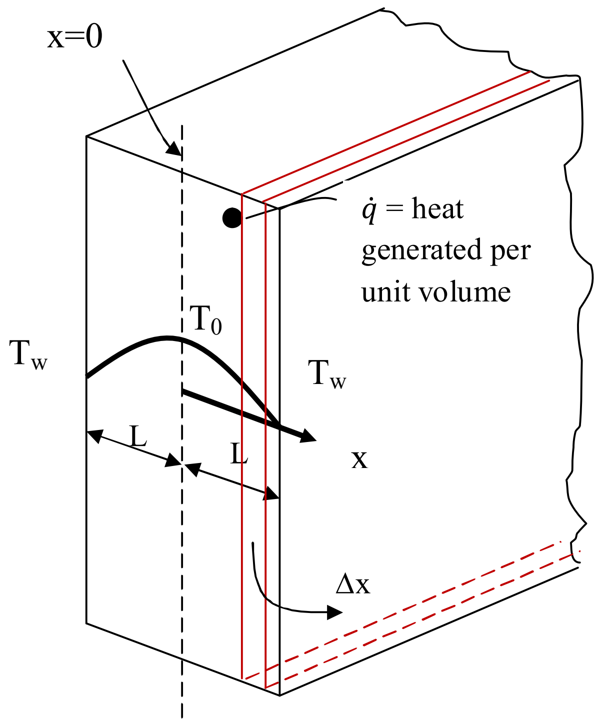

Figure 1 Sketch illustrating one-dimensional conduction problem with heat generation

Energy balance around shell

Rate of energy in - Rate of energy out + Rate of energy production = 0

$$A \cdot q_x|_x - A \cdot q_x|_{x+\Delta x} + (A\Delta x)\dot{q} = 0$$

Divided through equation by $A\Delta x$ and taking limit $\Delta x \to 0$ we obtain

$$-\lim_{\Delta x \to 0} \frac{(q_x)|_{x+\Delta x} - (q_x)|_x}{\Delta x} = -\dot{q}$$

The expression on the left side is the first derivative of q_x with respect to x, so that equation becomes

$$\frac{d}{dx}(q_x) = \dot{q}$$

The equation may be integrated to give

$$\int d(q_x) = \int \dot{q} dx$$

$$q_x = \dot{q}x + C_1 \qquad (1)$$

Apply Fourier's law: $q_x = -k\frac{dT}{dx}$

$$-k\frac{dT}{dx} = \dot{q}x + C_1$$

$$\int dT = \int \left(-\frac{\dot{q}}{k}x - \frac{C_1}{k}\right) dx$$

The equation may be integrated to give

$$T = -\frac{\dot{q}}{2k}x^2 - \frac{C_1}{k}x + C_2 \qquad (2)$$

BC1: $x = +L; \quad T = T_w$

BC2: $x = -L; \quad T = T_w$

BC3: $x = 0; \quad T = T_0$

Apply BC1 and BC2 into equation (2), respectively, we get

$$T_w = -\frac{\dot{q}}{2k}L^2 - \frac{C_1}{k}L + C_2 \qquad (3)$$

$$T_w = -\frac{\dot{q}}{2k}(-L)^2 - \frac{C_1}{k}(-L) + C_2 \qquad (4)$$

Subtract equation (4) by equation (3) we obtain

$$C_1 = 0 \qquad (5)$$

Substitute C_1 into equation (3) we get

$$C_2 = T_w + \frac{\dot{q}}{2k}L^2 \qquad (6)$$

Substitute C_1 and C_2 into the equation (2) we obtain the temperature distribution as

$$T = -\frac{\dot{q}}{2k}x^2 + T_w + \frac{\dot{q}}{2k}L^2 \qquad (7)$$

$$T = -\frac{\dot{q}}{2k}x^2 + T_w + \frac{\dot{q}}{2k}L^2$$

$$T - T_w = -\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}}{2k}L^2$$

The temperature distribution is therefore

$$T - T_w = \frac{\dot{q}L^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] \qquad (8)$$

<u>Alternative form of temperature distribution</u>

BC3: $x = 0;\quad T_0 = T_{max}\ or\ \frac{dT}{dx} = 0$

Apply BC1 and then BC3 into equation (2) we get

$$C_1 = 0 \qquad (9)$$

$$C_2 = T_0 \qquad (10)$$

Substitute C_1 and C_2 into the equation (2) we obtain the temperature distribution as

$$T - T_0 = -\frac{\dot{q}}{2k}x^2 \qquad (11)$$

Once the temperature and heat flux distributions are known, various information about the system may be obtained

(i) Maximum temperature rise (at x = 0)

$$T_{max} = T_0 \quad (12)$$

(ii) Average temperature rise

$$\overline{T} = \frac{T_{max}}{2} = \frac{T_0}{2} \quad (13)$$

(iii) Heat loss at the surface (for length L of plane)

$$Q_{loss} = 2Aq|_{x=L} = 2A(\dot{q}L) \quad (14)$$

Note: we got q from equation (1) and then substitute $C_1=0$