Example 9 Heat conduction with an electrical heat source

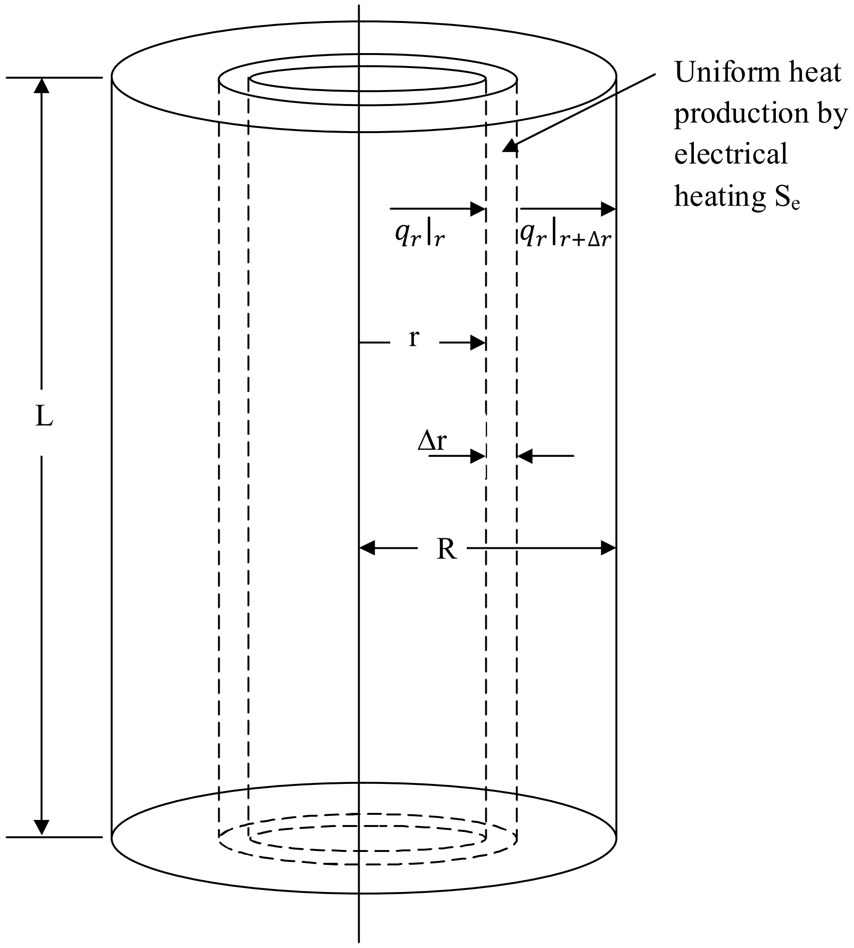

Figure 2 An electrically heated wire, showing the cylindrical shell over which the energy balance in made

BC1: at $r = 0, q_r$ is not infinite

BC2: at $r = R, T = T_0$

Energy balance around shell

Rate of energy in - Rate of energy out + Rate of energy production = 0

$$(2\pi rL)q_r|_r - (2\pi(r+\Delta r)L)q_r|_{r+\Delta r} + (2\pi r\Delta rL)S_e = 0$$

Approximate $\Delta r \ll r$ so $r + \Delta r \approx r$ we get

$$(2\pi rL)q_r|_r - (2\pi rL)q_r|_{r+\Delta r} + (2\pi r\Delta rL)S_e = 0$$

Divided through equation by $2\pi L\Delta r$ and taking limit $\Delta r \to 0$we obtain

$$-\lim_{\Delta r\to 0}\frac{(rq_r)|_{r+\Delta r} - (rq_r)|_r}{\Delta r} = -S_e r$$

The expression on the left side is the first derivative of rq_r with respect to r, so that equation becomes

$$\frac{d}{dr}(rq_r) = S_e r$$

$$\int d(rq_r) = \int S_e r dr + C_1$$

The equation may be integrated to give

$$rq_r = \frac{S_e r^2}{2} + C_1$$

$$q_r = \frac{S_e r}{2} + \frac{C_1}{r}$$

Apply BC1: at $r = 0, q_r$ is not infinite

$$q_r = \frac{S_e(0)}{2} + \frac{C_1}{(0)}$$

$$finite = finite + infinite$$

The term C_1 must be forced to be zero in order to make the right hand side is finite we get

$$q_r = \frac{S_e r}{2}$$

Apply Fourier's law: $q_r = -k\frac{dT}{dr}$

$$-k\frac{dT}{dr} = \frac{S_e r}{2}$$

Assume k is constant and indefinite integrate the equation we obtain

$$\int dT = -\int \frac{S_e r}{2k} dr$$

$$T = -\frac{S_e r^2}{4k} + C_2$$

Apply BC2: at $r = R, T = T_0$

$$T_0 = -\frac{S_e R^2}{4k} + C_2$$

$$C_2 = T_0 + \frac{S_e R^2}{4k}$$

Substitute C_2 into the equation so the equation becomes

$$T - T_0 = \frac{S_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Once the temperature and heat flux distributions are known, various information about the system may be obtained.

(i) Maximum temperature rise (at r = 0)

$$T_{max} - T_0 = \frac{S_e R^2}{4k}$$

(ii) Average temperature rise

$$\bar{T} - T_o = \frac{\int_0^{2\pi} \left(\int_0^R (T(r) - T_0) r dr \right) d\theta}{\int_0^{2\pi} \left(\int_0^R r dr \right) d\theta} = \frac{S_e R^2}{8k}$$

(iii) Heat outflow at the surface (for length L of wire)

$$Q|_{r=R} = 2\pi RL \cdot q_r|_{r=R} = 2\pi RL \cdot \frac{S_e R}{2} = \pi R^2 L \cdot S_e$$