Example 11 Heat conduction through composite walls

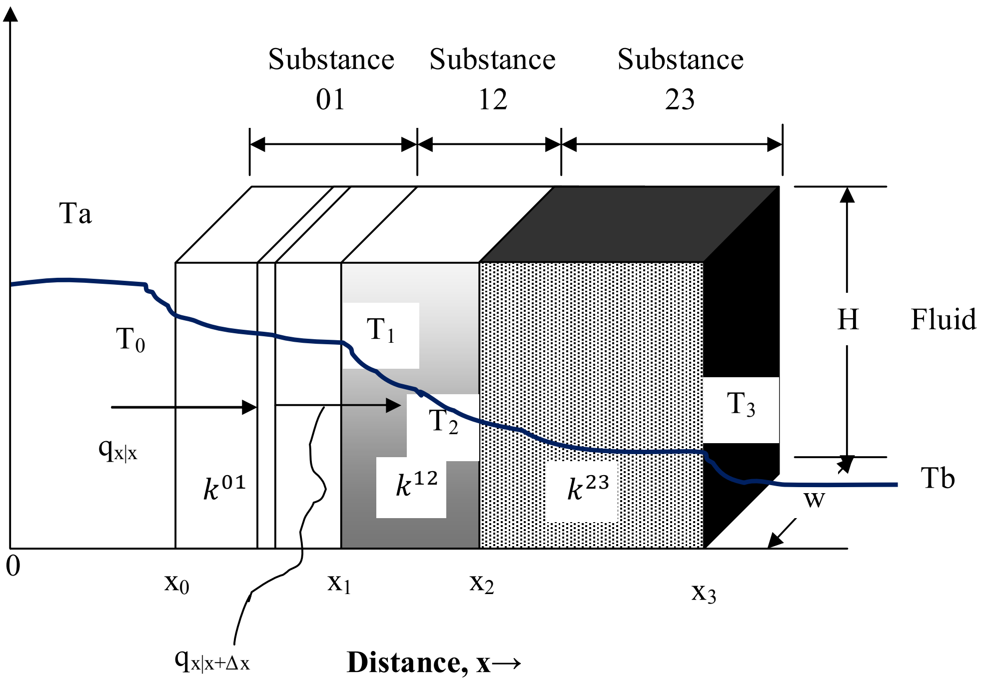

Figure 3 Heat conduction through a composite wall, located between two fluid streams at temperatures T_a and T_b

Energy balance around shell in region 01:

Rate of energy in - Rate of energy out + Rate of energy generation = 0

$$q_x|_x WH - q_x|_{x+\Delta x} WH = 0 \qquad (1)$$

Divided through the equation by WHΔx and taking limit Δx→0 we obtain

$$-\lim_{\Delta x \to 0} \frac{(q_x|_{x+\Delta x} - q_x|_x)}{\Delta x} = 0$$

The expression on the left side is the first derivative of q_x with respect to x, so that equation becomes

$$\frac{dq_x}{dx} = 0 \qquad (2)$$

Integration of this equation gives

$$q_x = q_0 \quad \text{(a constant)} \qquad (3)$$

The constant of integration, q_0, is the heat flux at the plane $x = x_0$. The development in eqn (1), (2) and (3) can be repeated for regions 12 and 23 with continuity conditions on q_x at interfaces, so that the heat flux is constant and the same for all three slabs:

Energy balance around regions 01, 12, 23:

$$q_x = q_0 \qquad (4)$$

Apply Fourier's law for each of the three regions and get

Region 01:

$$-k_{01} \frac{dT}{dx} = q_0 \qquad (5)$$

Region 02:

$$-k_{12}\frac{dT}{dx} = q_0 \qquad (6)$$

Region 03:

$$-k_{23}\frac{dT}{dx} = q_0 \qquad (7)$$

We now assume that k_{01}, k_{12}, k_{23} are constants. Then we integrate each equation over the entire thickness of the relevant slab of material to get

Region 01:

$$T_0 - T_1 = q_0\left(\frac{x_1 - x_0}{k_{01}}\right) \qquad (8)$$

Region 12:

$$T_1 - T_2 = q_0\left(\frac{x_2 - x_1}{k_{12}}\right) \qquad (9)$$

Region 23:

$$T_2 - T_3 = q_0\left(\frac{x_3 - x_2}{k_{23}}\right) \qquad (10)$$

In addition we have the two statements regarding the heat transfer at the surfaces according to Newton's law of cooling:

At surface 0:

$$T_a - T_0 = \frac{q_0}{h_0}$$

At surface 3:

$$T_3 - T_b = \frac{q_0}{h_3}$$

Addition of these last five equations then gives

$$T_a - T_b = q_0\left(\frac{1}{h_0} + \frac{x_1 - x_0}{k_{01}} + \frac{x_2 - x_1}{k_{12}} + \frac{x_3 - x_2}{k_{23}} + \frac{1}{h_3}\right)$$

or

$$q_0 = \frac{T_a - T_b}{\left(\frac{1}{h_0} + \frac{x_1 - x_0}{k_{01}} + \frac{x_2 - x_1}{k_{12}} + \frac{x_3 - x_2}{k_{23}} + \frac{1}{h_3}\right)}$$

$$q_0 = \frac{T_a - T_b}{\left(\frac{1}{h_0} + \sum_{j=1}^{n}\frac{x_j - x_{j-1}}{k_{j-1,j}} + \frac{1}{h_3}\right)}$$

Sometimes this result is rewritten in a form reminiscent of Newton's law of cooling, either in terms of the heat flux q_0 (J/m^2.s) or the heat flow Q_0(J/s):

$q_0 = U(T_a - T_b)$ or $Q_0 = U(WH)(T_a - T_b)$

The quantity U, called that "overall heat transfer coefficient," is given then by the following famous formula for the "additivity of resistances":

$$\frac{1}{U} = \frac{1}{h_0} + \sum_{j=1}^{n}\frac{x_j - x_{j-1}}{k_{j-1,j}} + \frac{1}{h_3}$$