Example 12 Heat conduction through composite cylindrical walls

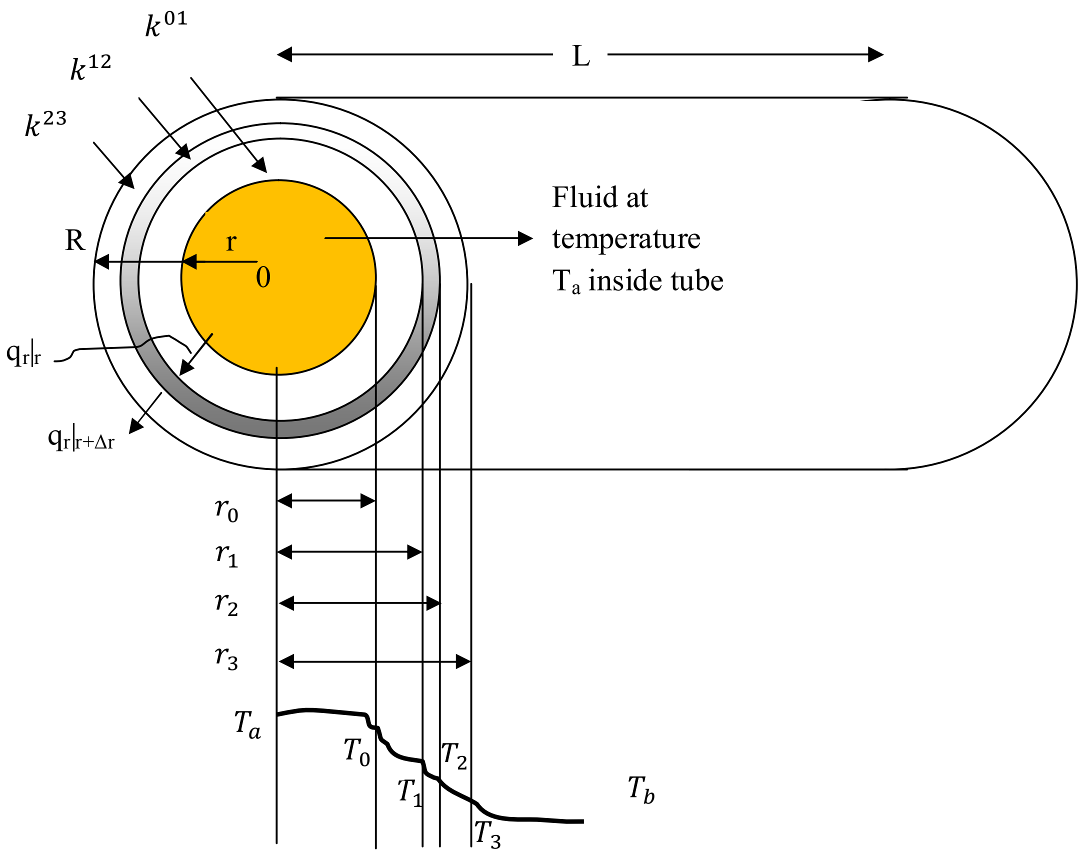

Figure 4 Heat conduction through a laminated tube with a fluid at temperature T_a inside the tube and temperature T_b outside

Energy balance on shell of volume $2\pi L\Delta r$ for region 01

Energy in shell – Energy out shell + Energy generated = 0

$$q_r|_r \cdot 2\pi r L - q_r|_{r+\Delta r} \cdot 2\pi(r+\Delta r)L = 0$$

Approximate $\Delta r \ll r$ so $r + \Delta r \approx r$ we get

$$q_r|_r \cdot 2\pi r L - q_r|_{r+\Delta r} \cdot 2\pi r L = 0$$

Dividing by $2\pi L\Delta r$ and taking the limit as Δr goes to zero gives

$$\frac{d}{dr}(r q_r) = 0$$

Integration of this equation gives

$$r q_r = r_0 q_0$$

in which r_0 is the inner radius of region 01, and q_0 is the heat flux there. In regions 12 and 23, $r q_r$ is equal to the same constant. Application of Fourier's law to the three regions gives

Region 01:

$$-k_{01} r \frac{dT}{dr} = r_0 q_0$$

Region 12:

$$-k_{12} r \frac{dT}{dr} = r_0 q_0$$

Region 23:

$$-k_{23} r \frac{dT}{dr} = r_0 q_0$$

If we assume that the thermal conductivities in the three annular regions are constants, then each of the above three equations can be integrated across its region to give

Region 01:

$$T_0 - T_1 = r_0 q_0 \frac{\ln(\frac{r_1}{r_0})}{k_{01}}$$

Region 12:

$$T_1 - T_2 = r_0 q_0 \frac{\ln(\frac{r_2}{r_1})}{k_{12}}$$

Region 23:

$$T_2 - T_3 = r_0 q_0 \frac{\ln(\frac{r_3}{r_2})}{k_{23}}$$

At the two fluid-solid interfaces we can write Newton's law of cooling:

surface 0:

$$T_a - T_0 = \frac{q_0}{h_0}$$

Surface 3:

$$T_3 - T_b = \frac{q_3}{h_3} = \frac{q_0}{h_3}\frac{r_0}{r_3}$$

Addition of the preceding five equations gives an equation for $T_a - T_b$. Then the equation is solved for q_0 to give

$$Q_0 = 2\pi L r_0 q_0 = \frac{2\pi L(T_a - T_b)}{\left(\frac{1}{r_0 h_0} + \frac{\ln{(r_1/r_0)}}{k_{01}} + \frac{\ln{(r_2/r_1)}}{k_{12}} + \frac{\ln{(r_3/r_2)}}{k_{23}} + \frac{1}{r_3 h_3}\right)}$$

We now define an "overall heat transfer coefficient based on the inner surface" U_i by

$$Q_0 = 2\pi L r_0 q_0 = U_i(2\pi L r_0)(T_a - T_0)$$

$$U_i = \frac{2\pi L r_0 q_0}{(2\pi L r_0)(T_a - T_0)}$$

$$U_i = \frac{1}{(2\pi L r_0)(T_a - T_0)} \frac{2\pi L (T_a - T_b)}{\left(\frac{1}{r_0 h_0} + \frac{\ln{(r_1/r_0)}}{k_{01}} + \frac{\ln{(r_2/r_1)}}{k_{12}} + \frac{\ln{(r_3/r_2)}}{k_{23}} + \frac{1}{r_3 h_3}\right)}$$

$$U_i = \frac{1}{r_0(T_a - T_0)} \frac{(T_a - T_b)}{\left(\frac{1}{r_0 h_0} + \frac{\ln{(r_1/r_0)}}{k_{01}} + \frac{\ln{(r_2/r_1)}}{k_{12}} + \frac{\ln{(r_3/r_2)}}{k_{23}} + \frac{1}{r_3 h_3}\right)}$$

We now define an "overall heat transfer coefficient based on the outer surface" U_0 by

$$Q_0 = 2\pi L r_0 q_0 = U_0(2\pi L r_3)(T_3 - T_b)$$

$$U_0 = \frac{2\pi L r_0 q_0}{(2\pi L r_3)(T_3 - T_b)}$$

$$U_0 = \frac{1}{(2\pi L r_3)(T_3 - T_b)} \frac{2\pi L (T_a - T_b)}{\left(\frac{1}{r_0 h_0} + \frac{\ln{(r_1/r_0)}}{k_{01}} + \frac{\ln{(r_2/r_1)}}{k_{12}} + \frac{\ln{(r_3/r_2)}}{k_{23}} + \frac{1}{r_3 h_3}\right)}$$

$$U_0 = \frac{(T_a - T_b)}{(T_3 - T_b)\left(\frac{r_3}{r_0 h_0} + \frac{r_3 \ln{(r_1/r_0)}}{k_{01}} + \frac{r_3 \ln{(r_2/r_1)}}{k_{12}} + \frac{r_3 \ln{(r_3/r_2)}}{k_{23}} + \frac{1}{h_3}\right)}$$