Example 13 Composite Sphere

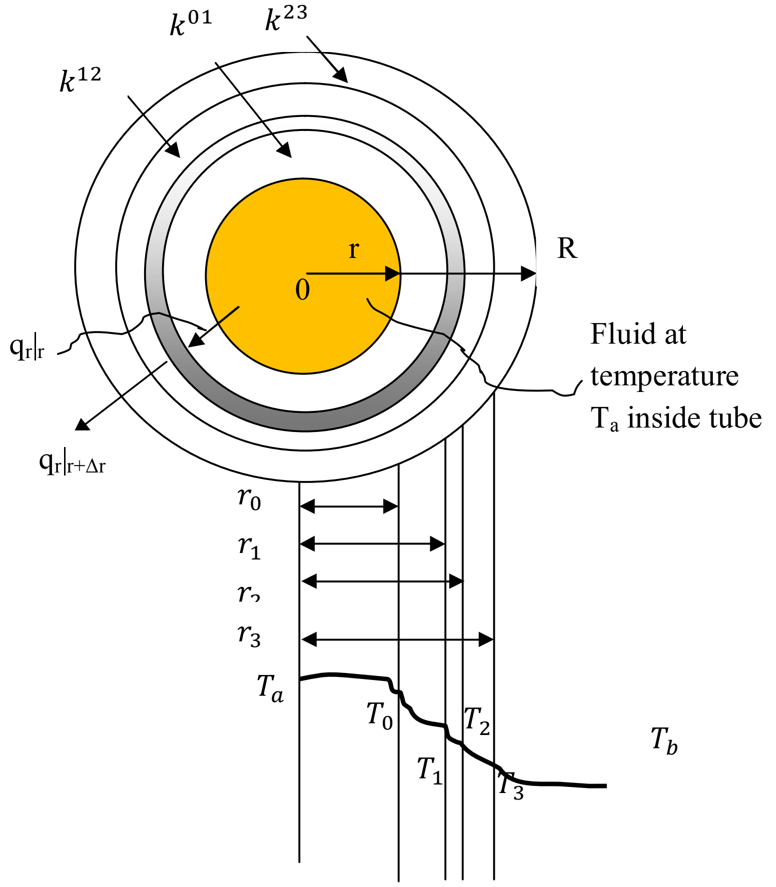

Figure 4 Heat conduction through a laminated tube with a fluid at temperature T_a inside the tube and temperature T_b outside

Energy balance on shell of volume $4\pi r \Delta r$ for region 01

Energy in shell – Energy out shell + Energy generated = 0

$$q_r|_r \cdot 4\pi r^2 - q_r|_{r+\Delta r} \cdot 4\pi(r+\Delta r)^2 = 0$$

Approximate $\Delta r \ll r$ so $r + \Delta r \approx r$ we get

$$q_r|_r \cdot 4\pi r^2 - q_r|_{r+\Delta r} \cdot 4\pi r^2 = 0$$

Dividing by $4\pi\Delta r$ and taking the limit as Δr goes to zero gives

$$\frac{d}{dr}(r^2 q_r) = 0$$

Integration of this equation gives

$$r^2 q_r = r_0^2 q_0$$

in which r_0 is the inner radius of region 01, and q_0 is the heat flux there. In regions 12 and 23, $r^2 q_r$ is equal to the same constant. Application of Fourier's law to the three regions gives

Region 01:

$$-k_{01} r^2 \frac{dT}{dr} = r_0^2 q_0$$

Region 12:

$$-k_{12} r^2 \frac{dT}{dr} = r_0^2 q_0$$

Region 23:

$$-k_{23} r^2 \frac{dT}{dr} = r_0^2 q_0$$

If we assume that the thermal conductivities in the three annular regions are constants, then each of the above three equations can be integrated across its region to give

Region 01:

$$T_0 - T_1 = \frac{r_0^2 q_0}{k_{01}}\left(\frac{1}{r_0} - \frac{1}{r_1}\right)$$

Region 12:

$$T_1 - T_2 = \frac{r_0^2 q_0}{k_{12}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Region 23:

$$T_2 - T_3 = \frac{r_0^2 q_0}{k_{23}}\left(\frac{1}{r_2} - \frac{1}{r_3}\right)$$

At the two fluid-solid interfaces we can write Newton's law of cooling:

Surface 0:

$$T_a - T_0 = \frac{q_0}{h_0}$$

Surface 3:

$$T_3 - T_b = \frac{q_3}{h_3} = \frac{q_0}{h_3}\frac{r_0^2}{r_3^2}$$

Addition of the preceding five equations gives an equation for $T_a - T_b$. Then the equation is solved for q_0 to give

$$Q_0 = 4\pi r_0^2 q_0 = \frac{4\pi(T_a - T_b)}{\left(\frac{1}{r_0^2 h_0} + \frac{1}{k_{01}}\left(\frac{1}{r_0} - \frac{1}{r_1}\right) + \frac{1}{k_{12}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{1}{k_{23}}\left(\frac{1}{r_2} - \frac{1}{r_3}\right) + \frac{1}{r_3^2 h_3}\right)}$$

We now define an "overall heat transfer coefficient based on the inner surface" U_i by

$$Q_0 = 4\pi r_0^2 q_0 = U_i(4\pi r_0^2)(T_a - T_0)$$

$$U_i = \frac{4\pi r_0^2 q_0}{(4\pi r_0^2)(T_a - T_0)}$$

$$U_i = \frac{1}{(4\pi r_0^2)(T_a - T_0)} \frac{4\pi(T_a - T_b)}{\left(\frac{1}{r_0^2 h_0} + \frac{1}{k_{01}}(\frac{1}{r_0} - \frac{1}{r_1}) + \frac{1}{k_{12}}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{1}{k_{23}}(\frac{1}{r_2} - \frac{1}{r_3}) + \frac{1}{r_3^2 h_3}\right)}$$

$$U_i = \frac{1}{(T_a - T_0)} \frac{(T_a - T_b)}{\left(\frac{1}{h_0} + \frac{r_0^2}{k_{01}}(\frac{1}{r_0} - \frac{1}{r_1}) + \frac{r_0^2}{k_{12}}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{r_0^2}{k_{23}}(\frac{1}{r_2} - \frac{1}{r_3}) + \frac{r_0^2}{r_3^2 h_3}\right)}$$

We now define an "overall heat transfer coefficient based on the outer surface" U_0 by

$$Q_0 = 4\pi r_3^2 q_0 = U_0(4\pi r_3^2)(T_3 - T_b)$$

$$U_0 = \frac{4\pi r_3^2 q_0}{(4\pi r_3^2)(T_3 - T_b)}$$

$$U_0 = \frac{1}{(4\pi r_3^2)(T_3 - T_b)} \frac{4\pi(T_a - T_b)}{\left(\frac{1}{r_0^2 h_0} + \frac{1}{k_{01}}(\frac{1}{r_0} - \frac{1}{r_1}) + \frac{1}{k_{12}}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{1}{k_{23}}(\frac{1}{r_2} - \frac{1}{r_3}) + \frac{1}{r_3^2 h_3}\right)}$$

$$U_0 = \frac{1}{(r_3^2)(T_3 - T_b)} \frac{(T_a - T_b)}{\left(\frac{1}{r_0^2 h_0} + \frac{1}{k_{01}}(\frac{1}{r_0} - \frac{1}{r_1}) + \frac{1}{k_{12}}(\frac{1}{r_1} - \frac{1}{r_2}) + \frac{1}{k_{23}}(\frac{1}{r_2} - \frac{1}{r_3}) + \frac{1}{r_3^2 h_3}\right)}$$