Example 14 Heat conduction in a cooling Fin

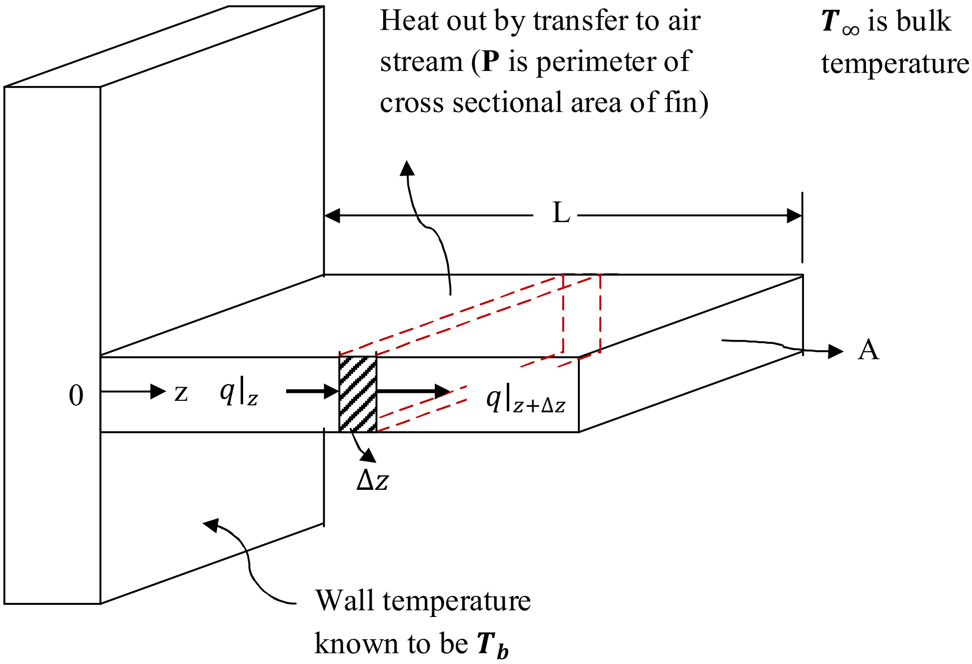


Figure 7 A simple cooling Fin

Conduction heat balance around shell at steady state

[Rate of heat in] – [Rate of heat out] - [Rate of heat loss] = 0

$$q|_z \cdot A - q|_{z+\Delta z} \cdot A - h(T - T_\infty)P\Delta z = 0$$

Division by $A\Delta z$ and taking the limit as Δz approaches zero gives

$$-\lim_{\Delta z \to 0}\left(\frac{q|_{z+\Delta z} - q|_z}{\Delta z}\right) = \frac{hP}{A}(T - T_\infty)$$

$$-\frac{dq}{dz} = \frac{hP}{A}(T - T_\infty)$$

We now insert Fourier's law ($q_z = -kdT/dz$), in which k is the thermal conductivity of the metal. If we assume that k is constant, we then get

$$-\frac{d}{dz}\left(-k\frac{dT}{dz}\right) = \frac{hP}{A}(T - T_\infty)$$

$$\frac{d^2T}{dz^2} = \frac{hP}{kA}(T - T_\infty) \qquad (1)$$

Set dimensionless group: $x = z/L$; $\theta = \frac{T-T_\infty}{T_b-T_\infty}$ we obtain

$$z = xL;\ \ d\theta = \frac{dT}{(T_b - T_\infty)}$$

$$z^2 = (xL)^2$$

$$dz^2 = L^2 dx^2;\ d^2\theta = \frac{d^2T}{(T_b - T_\infty)}$$

Equation (1) becomes

$$\frac{(T_b - T_\infty)d^2\theta}{L^2 dx^2} = \frac{hP}{kA}(T - T_\infty)$$

$$\frac{d^2\theta}{dx^2} = \frac{hPL^2}{kA}\frac{(T - T_\infty)}{(T_b - T_\infty)}$$

$$\frac{d^2\theta}{dx^2} = \frac{hPL^2}{kA}\theta$$

Set $\eta^2 = \frac{L^2 hP}{kA}$ we obtain

$$\frac{d^2\theta}{dx^2} - \eta^2\theta = 0 \qquad (2)$$

Method of characteristic solution: $\theta = e^{sx}$

$$\frac{d\theta}{dx} = se^{sx}$$

$$\frac{d^2\theta}{dx^2} = \frac{d^2se^{sx}}{dx^2} = s^2e^{sx}$$

Equation (2) becomes

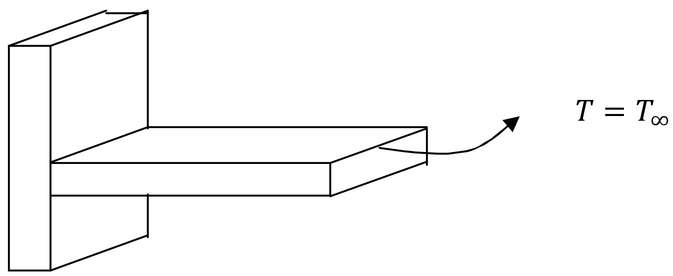
$$s^2e^{sx} - \eta^2e^{sx} = 0$$

$$s = \pm\eta$$

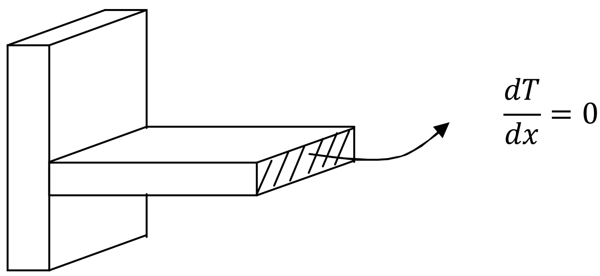
$$\boxed{\theta = C_1e^{\eta x} + C_2e^{-\eta x} \qquad (3)}$$

The boundary conditions depend on the physical situation. Several cases may be considered:

CASE 1 The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

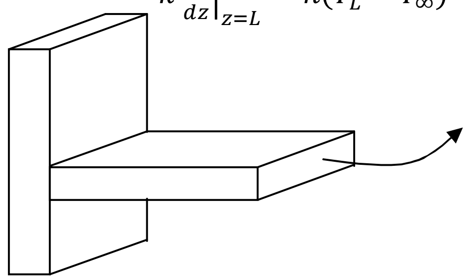


CASE 2 The end of the fin is insulated so that $\frac{dT}{dz} = 0$ at $z = L$.



CASE 3 The fin is of finite length and loses heat by convection from its end.

$$-k\left.\frac{dT}{dz}\right|_{z=L} = h(T_L - T_\infty)$$



<u>Case I</u>: The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

Assume long fin; $L \to \infty$

BC1: $z = 0, \quad x = 0, \quad T = T_b, \quad \theta = 1$

BC2: $z = L = \infty, \; x = \infty, \; T = T_\infty, \quad \theta = 0$

Insert BC2 into equation (3)

$$0 = C_1 e^{\infty} + C_2 e^{-\infty}$$

$$finite = infinite + finite$$

We have to force term $C_1 = 0$ in order to valid the above equation

Equation (3) becomes

$$\theta = C_2 e^{-\eta x} \qquad (4)$$

Use BC1 substitute into equation (4)

$$1 = C_2 e^{0}$$

$$C_2 = 1$$

Substitute C_2 into equation (4) we get

$$\theta = e^{-\eta x} \qquad (5)$$

Substitute $\Theta = \frac{T - T_\infty}{T_b - T_\infty}$, $\eta^2 = \frac{L^2 hP}{kA}$ and $x = \frac{z}{L}$ to equation (5) we obtain

$$\frac{T - T_\infty}{T_b - T_\infty} = e^{-L\sqrt{\frac{hP}{kA}}\frac{z}{L}}$$

We obtain the temperature profile as

$$T = T_\infty + (T_b - T_\infty)exp\left[-z\left(\frac{hP}{kA}\right)^{1/2}\right]$$

We can determine *heat loss* by performing derivatives with respect to z of temperature profile

$$Q_{loss} = A(-k)\frac{dT}{dz}\bigg|_{z=0}$$

$$\frac{dT}{dz} = (T_b - T_\infty)exp\left[-z\left(\frac{hP}{kA}\right)^{1/2}\right]\left(-\left(\frac{hP}{kA}\right)^{1/2}\right)$$

$$A(-k)\frac{dT}{dz}\bigg|_{z=0} = A(-k)(T_b - T_\infty)\left(-\sqrt{\frac{hP}{kA}}\right)$$

$$Q_{loss} = kA(T_b - T_\infty)\sqrt{\frac{hP}{kA}} = (T_b - T_\infty)\sqrt{hPkA}$$

$$Q_{loss} = (T_b - T_\infty)\sqrt{hPkA}$$

We can determine *fin efficiency* (η_f) which is defined by

$$\eta_f = \frac{actual\ rate\ of\ heat\ loss\ from\ the\ fin}{rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_b}$$

$$Q_{loss} = (T_b - T_\infty)\sqrt{hPkA}$$

$$Q_{Ideal} = h(T_b - T_\infty)PL$$

$$\eta_f = \frac{Q_{actual}}{Q_{Ideal}} = \frac{(T_b - T_\infty)\sqrt{hPkA}}{(T_b - T_\infty)hPL}$$

$$\eta_f = \sqrt{\frac{kA}{hPL^2}}$$

We can determine *fin effectiveness* (ε_f)

$$\varepsilon_f = \frac{actual\ rate\ of\ heat\ loss\ from\ the\ fin}{rate\ of\ heat\ loss\ without\ fin}$$

$$Q_{loss} = (T_b - T_\infty)\sqrt{hPkA}$$

$$Q_{no\ fin} = h(T_b - T_\infty)A$$

$$\varepsilon_f = \frac{Q_{actual}}{Q_{no\ fin}} = \frac{(T_b - T_\infty)\sqrt{hPkA}}{(T_b - T_\infty)hA}$$

$$\varepsilon_f = \sqrt{\frac{Pk}{hA}}$$

<u>Case II</u>: The end of fin is insulated so that $\frac{dT}{dx} = 0$ at x=L **(No heat is lost from the end or from the edges.)**

From equation (3);

$$\theta = C_1 e^{\eta x} + C_2 e^{-\eta x} \qquad (3)$$

From

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$2\sinh x = e^x - e^{-x}$$

$$e^x = 2\sinh x + e^{-x} \qquad (6)$$

From

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$2\cosh x = e^x + e^{-x}$$

$$e^x = 2\cosh x - e^{-x} \qquad (7)$$

Set equation (6) equal to equation (7) we obtain

$$2\sinh x + e^{-x} = 2\cosh x - e^{-x}$$

$$2e^{-x} = 2\cosh x - 2\sinh x$$

$$e^{-x} = \cosh x - \sinh x \qquad (8)$$

Substitute equation (8) into (6) we obtain

$$e^x = 2\sinh x + \cosh x - \sinh x$$

$$e^x = \sinh x + \cosh x \qquad (9)$$

Substitute equation (8) and (9) into the equation (3), we get

$$\theta = C_1[\sinh \eta x + \cosh \eta x] + C_2[\cosh \eta x - \sinh \eta x]$$

$$\theta = \sinh \eta x \underbrace{[C_1 - C_2]}_{\text{A}} + \cosh \eta x \underbrace{[C_1 + C_2]}_{\text{B}}$$

$$\theta = A \sinh \eta x + \text{B}\cosh \eta x \qquad (10)$$

BC1: $z = 0;\quad x = 0;\quad T = T_b;\qquad \theta = 1$

BC2: $z = L;\quad x = 1;\quad \left.\frac{dT}{dz}\right|_{z=L} = 0;\qquad \frac{(T-T_\infty)d\theta}{Ldx} = \frac{d\theta}{dx} = 0$

Use BC1 substitute into (10);

$$1 = A \sinh(0) + \text{B}\cosh(0)$$

$$B = 1$$

$$\theta = A \sinh \eta x + \cosh \eta x \qquad (11)$$

$$\frac{d\theta}{dx} = A\,\eta\cosh(\eta x) + \eta\sinh(\eta x) \qquad (12)$$

Use BC2 substitute into equation 12;

$$0 = A(1)\cosh(\eta) + \sinh(\eta)$$

$$A = -\frac{\sinh \eta}{\cosh \eta} = -\tanh \eta$$

Equation (12) becomes

$$\theta = -\tanh \eta \cdot \sinh(\eta x) + \cosh(\eta x)$$

$$\frac{T - T_\infty}{T_b - T_\infty} = -\tanh\eta \cdot \sinh(\eta x) + \cosh(\eta x)$$

$$T = T_\infty + (T_b - T_\infty)[-\tanh\eta \cdot \sinh(\eta x) + \cosh(\eta x)]$$

Substitute $\eta = L\sqrt{\frac{hP}{kA}}$ into the equation

$$T = T_\infty + (T_b - T_\infty)\left[-\tanh\eta \cdot \sinh\left(L\sqrt{\frac{hP}{kA}} \cdot \frac{z}{L}\right) + \cosh\left(L\sqrt{\frac{hP}{kA}} \cdot \frac{z}{L}\right)\right]$$

$$T = T_\infty + (T_b - T_\infty)\left[-\tanh\eta \cdot \sinh\left(z\sqrt{\frac{hP}{kA}}\right) + \cosh\left(z\sqrt{\frac{hP}{kA}}\right)\right]$$

We can determine *heat loss* by performing derivatives with respect to z of temperature profile

$$Q_{loss} = A(-k)\left.\frac{dT}{dz}\right|_{z=0}$$

$$\left.\frac{dT}{dz}\right|_{z=0} = (T_b - T_\infty)\left[-\sqrt{\frac{hP}{kA}} \cdot \tanh\eta \cdot \cosh(0) + \sqrt{\frac{hP}{kA}}\sinh(0)\right]$$

$$\left.\frac{dT}{dz}\right|_{z=0} = (T_b - T_\infty)\left[-\sqrt{\frac{hP}{kA}} \cdot \tanh\eta \cdot (1) + 0\right]$$

$$Q_{loss} = A \cdot (-k)\left.\frac{dT}{dz}\right|_{z=0} = -(T_b - T_\infty)\sqrt{\frac{hP}{kA}} \cdot \tanh\eta\,(-k)A$$

We can express the term $\sqrt{\frac{hP}{kA}}$ into η by multiplying $\sqrt{\frac{kA}{hP}} \cdot \sqrt{\frac{hP}{kA}}$ to above equation.

$$Q_{loss} = (T_b - T_\infty)kA\tanh\eta\sqrt{\frac{hP}{kA}} \cdot \sqrt{\frac{kA}{hP}} \cdot \sqrt{\frac{hP}{kA}}$$

$$Q_{loss} = (T_b - T_\infty)kA\frac{\tanh\eta}{\eta}\frac{hP}{kA}\sqrt{\frac{kA}{hP}} \cdot \frac{1}{L}$$

(Remark: $\eta = \sqrt{\frac{kA}{hP}} \cdot \frac{1}{L}$)

$$Q_{loss} = (T_b - T_\infty)\frac{hPL\tanh\eta}{\eta}$$

We can determine *fin efficiency* (η_f) which is defined by

$$\eta_f = \frac{Q_{actual}}{Q_{Ideal}}$$

$$Q_{loss} = (T_b - T_\infty)\frac{hPL\tanh\eta}{\eta}$$

$$Q_{Ideal} = (T_b - T_\infty)hPL$$

$$\eta_f = \frac{(T_b - T_\infty)hPL\tanh\eta}{\eta} \cdot \frac{1}{(T_b - T_\infty)hPL}$$

$$\eta_f = \frac{\tanh\eta}{\eta}$$

We can determine *fin effectiveness* (ε_f)

$$\epsilon_f = \frac{Q_{loss}}{Q_{no\ fin}}$$

$$Q_{loss} = (T_b - T_\infty)\frac{hPL\tanh\eta}{\eta}$$

$$Q_{no\ fin} = (T_b - T_\infty)hA$$

$$\epsilon_f = \frac{(T_b - T_\infty)hPL\tanh\eta}{\eta} \cdot \frac{1}{(T_b - T_\infty)kA}$$

$$\epsilon_f = \frac{PL\tanh\eta}{A\eta}$$

<u>Case III</u>: The fin is of finite length and loses heat by convection from its end. (Facing convection BC at the tip of fin)

From equation (10)

$$\theta = A\sinh\eta x + \mathrm{B}\cosh\eta x \qquad (10)$$

BC1: $z = 0;\ \ x = 0,\ \ T = T_b;\ \ \theta = 1$

BC2: $z = L;\ \ x = 1,\ \ T = T_L;\ -k\frac{dT}{dz}\Big|_{z=L} = h(T_b - T_\infty)$

Turn the BC2 into dimensionless term

$$\frac{dT}{dz} = \frac{(T_b - T_\infty)d\theta}{Ldx}$$

$$-k\frac{dT}{dz}\Big|_{z=L} = -\frac{k(T_b - T_\infty)}{L}\frac{d\theta}{dx}\Big|_{x=1} = h(T_b - T_\infty)$$

We get

$$\frac{d\theta}{dx}\Big|_{x=1} = -\frac{Lh}{k}\theta|_{x=1}$$

Use BC1:

$$1 = A(0) + B(1)$$

$$B = 1$$

$$\theta = A\sinh\eta x + \cosh\eta x \qquad (14)$$

$$\theta|_{x=1} = A\sinh\eta + \cosh\eta \qquad (15)$$

$$\frac{d\theta}{dx}\Big|_{x=1} = A\eta\cosh\eta + \eta\sinh\eta \qquad (16)$$

From BC2: $\left.\frac{d\theta}{dx}\right|_{x=1} = \left.\frac{-Lh\theta}{k}\right|_{x=1}$, substitute equation (15) and (16) into the BC2.

$$\left.\frac{d\theta}{dx}\right|_{x=1} = \left.\frac{-Lh\theta}{k}\right|_{x=1}$$

$$A\eta\cosh\eta + \eta\sinh\eta = \frac{-Lh}{k}[A\sinh\eta + \cosh\eta]$$

$$A\left[\eta\cosh\eta + \frac{Lh}{k}\sinh\eta\right] = \frac{-Lh}{k}\cosh\eta - \eta\sinh\eta$$

$$A = \frac{-(\eta\sinh\eta + \frac{Lh}{k}\cosh\eta)}{(\eta\cosh\eta + \frac{Lh}{k}\sinh(\eta))}$$

$$\theta = \cosh\eta x - \left[\frac{\eta\sinh\eta + \frac{Lh}{k}\cosh\eta}{\eta\cosh\eta + \frac{Lh}{k}\sinh\eta}\right]\sinh\eta x$$

Temperature profile

$$T = T_\infty + (T_b - T_\infty)\left[\cosh\left(z\sqrt{\frac{hP}{kA}}\right) + A'\sinh\left(z\sqrt{\frac{hP}{kA}}\right)\right]$$

and $A' = \frac{-(\eta\sinh\eta + \frac{Lh}{k}\cosh\eta)}{(\eta\cosh\eta + \frac{Lh}{k}\sinh(\eta))}$

We can determine *heat loss* by performing derivatives with respect to z of temperature profile

$$\left.\frac{dT}{dz}\right|_{z=0} = (T_b - T_\infty)\left[\sqrt{\frac{hP}{kA}}\sinh(0) + A'\sqrt{\frac{hP}{kA}}\cosh(0)\right]$$

$$A\cdot(-k)\left.\frac{dT}{dz}\right|_{z=0} = (T_b - T_\infty)\sqrt{\frac{hP}{kA}}(-k)A = Q_{loss}$$

Substitute $A' = \frac{-\left(\eta \sinh \eta + \frac{Lh}{k} \cosh \eta\right)}{\eta \cosh \eta + \frac{Lh}{k} \sinh \eta}$, we obtain

$$Q_{loss} = (T_b - T_\infty)\left[\frac{-\left(\eta \sinh \eta + \frac{Lh}{k} \cosh \eta\right)}{\eta \cosh \eta + \frac{Lh}{k} \sinh \eta}\right]\sqrt{\frac{hP}{kA}}(-k)A$$

(Remark: $\eta = L\sqrt{\frac{Ph}{kA}}$)

$$Q_{loss} = (T_b - T_\infty)\frac{\eta kA}{L}\underbrace{\left[\frac{\eta \sinh \eta + \frac{Lh}{k} \cosh \eta}{\eta \cosh \eta + \frac{Lh}{k} \sinh \eta}\right]}_{\text{M}}$$

$$Q_{loss} = (T_b - T_\infty)\eta \frac{kA}{L} M$$

$$Q_{Ideal} = (T_b - T_\infty)hPL$$

We can determine *fin efficiency* (η_f) which is defined by

$$\eta_f = \frac{M}{\eta}$$

We can determine *fin effectiveness* (ε_f) which is defined by

$$Q_{no\ fin} = (T_b - T_\infty)hA$$

$$\varepsilon_f = \frac{\eta kAM}{L}\frac{1}{hA}$$

$$\varepsilon_f = \frac{1}{hA}\sqrt{\frac{Pk}{hA}}M$$