Example 15 Heat conduction in a cooling Fin

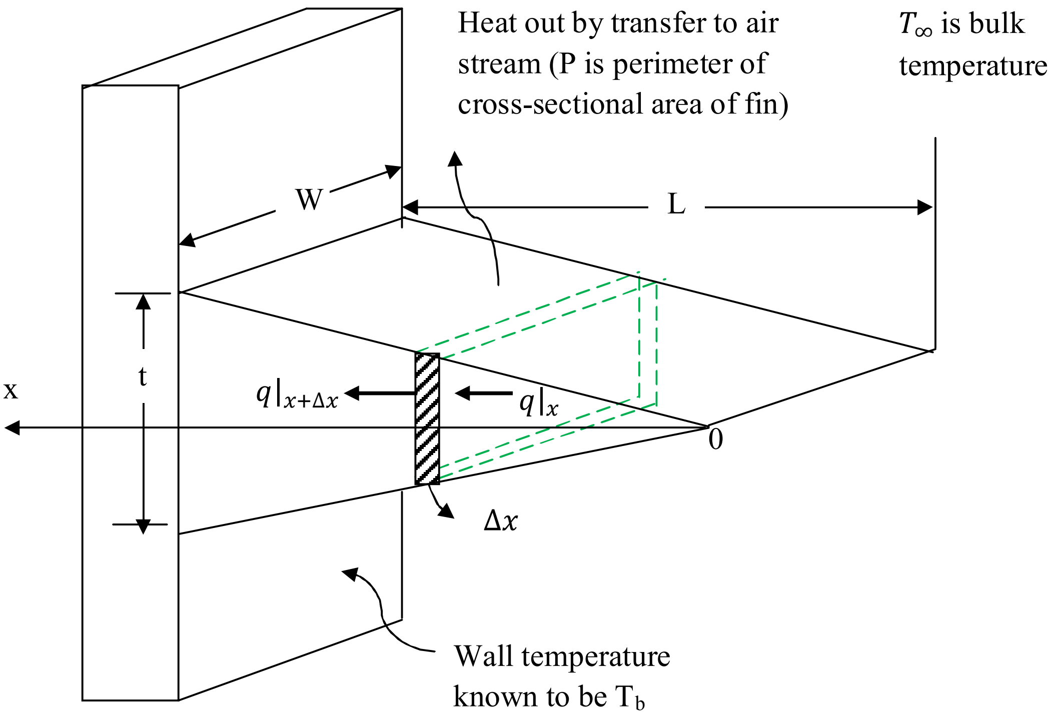


Figure 8 A simple cooling fin

Assume:

$$\frac{x}{L} = \frac{A_x}{A_L} = \frac{P_x}{P_L}$$

$$A_x = \frac{A_L}{L}x; P_x = \frac{P_L}{L}x$$

Conduction heat balance around shell at steady state

[Rate of heat in] – [Rate of heat out] - [Rate of heat loss] = 0

$$q|_x \cdot A_x - q|_{x+\Delta x} \cdot A_x - h(T - T_\infty)P_x \Delta x = 0$$

Substitute $A_x = \frac{A_L}{L}x$ and $P_x = \frac{P_L}{L}x$ into the above equation

$$q|_x \cdot \frac{A_L}{L}x - q|_{x+\Delta x} \cdot \frac{A_L}{L}x - h(T - T_\infty)\frac{P_L}{L}x\Delta x = 0$$

Divided by $\frac{A_L}{L}\Delta x$ through equation and taking limit $\Delta x \to 0$ we get

$$-\lim_{\Delta x \to 0}\left(\frac{q|_{x+\Delta x} \cdot x - q|_x \cdot x}{\Delta x}\right) - h(T - T_\infty)\frac{P_L}{A_L}x = 0$$

From the definition of derivative equation we obtain

$$-\frac{\mathrm{d}(x \cdot q_x)}{\mathrm{dx}} - h(T - T_\infty)\frac{P_L}{A_L}x = 0$$

$$-\frac{\mathrm{d}(x \cdot q_x)}{\mathrm{dx}} = h(T - T_\infty)\frac{P_L}{A_L}x$$

[Consider the ratio between top surface area (P_L) and cross-sectional area(A_L) we get

$$\frac{P_L}{A_L} = \frac{2(w + t)}{tw} = \frac{2}{t} + \frac{2}{w}$$

If w is very large, we can neglect the term $\frac{2}{w} \approx 0$ we get $\frac{P_L}{A_L} = \frac{2}{t}$

$$-\frac{1}{\mathrm{x}} \cdot \frac{\mathrm{d}(x \cdot q_x)}{\mathrm{dx}} = h(T - T_\infty)\frac{P_L}{A_L} \qquad (1)$$

Apply Fourier's law;

$$\frac{\mathrm{k}}{\mathrm{x}}\frac{\mathrm{d}}{\mathrm{dx}}\left(x\frac{dT}{dx}\right) = h(T - T_\infty)\frac{P_L}{A_L}$$

$$\frac{1}{x}\frac{d}{dx}\left(\frac{xdT}{dx}\right) = (T - T_\infty)\frac{P_L h}{kA_L} \qquad (2)$$

Set dimensionless group: $\theta = \frac{(T-T_\infty)}{(T_b - T_\infty)}$: $dT = (T_b - T_\infty)d\theta$, equation (2) becomes:

$$d\theta = d\left[\frac{(T - T_\infty)}{(T_b - T_\infty)}\right]$$

$$d\theta = \frac{1}{(T_b - T_\infty)}d[(T - T_\infty)]$$

$$d\theta = \frac{1}{(T_b - T_\infty)}[dT + 0]$$

$$(T_b - T_\infty)d\theta = dT$$

$$\frac{1}{x}\frac{d}{dx}\left(\frac{xd\theta}{dx}\right) = \frac{(T - T_\infty)}{(T_b - T_\infty)}\frac{P_L h}{A_L k}$$

$$\frac{1}{x}\frac{d}{dx}\left(\frac{xd\theta}{dx}\right) = \theta\frac{P_L h}{A_L k}$$

$$\frac{d}{dx}\left(x \cdot \frac{d\theta}{dx}\right) = \theta x\frac{P_L h}{A_L k}$$

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - \theta x\left(\frac{P_L h}{A_L k}\right) = 0$$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - x^2\theta\left(\frac{P_L h}{A_L k}\right) = 0$$

Set: $\frac{P_L h}{A_L k} = \beta^2$;

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - x^2\beta^2\theta = 0$$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - (x^2\beta^2 - 0^2)\theta = 0 \qquad (3)$$

From Modified Bessel:

$$x^2 \frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - (x^2\beta^2 + n^2)\theta = 0$$

Solution

$$\theta = AI_0(\beta x) + BK_0(\beta x) \qquad (4)$$

<u>Case I</u>: The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

Assume long fin; $L \to \infty$

BC1: $x = 0; T = T_\infty : \theta = 0$

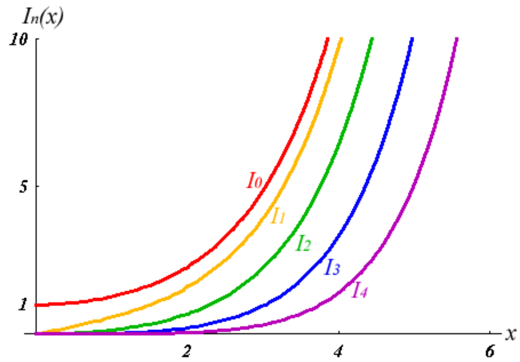
BC2: $x = L; T = T_b : \theta = 1$

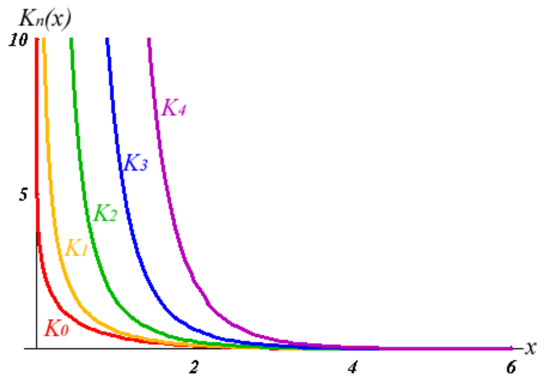
Use BC1 to substitute into eqn (4);

$$\underset{\text{Finite} \;=}{0} = \underbrace{AI_0(0)}_{A\,(1)} + \underbrace{BK_0(0)}_{\infty} \qquad (4)$$

Because term $K_0(0) = \infty$ and the equation is finite, we have to force B = 0 to valid the equation. Eqn (4) becomes:

$$\theta = AI_0(\beta x) \qquad (5)$$





$$\frac{dI_0(x)}{dx} = I_1(x)$$

$$\frac{dK_0(x)}{dx} = -K_1(x)$$

Use BC2 substitute into equation (5)

$$1 = AI_0(\beta L)$$

$$A = \frac{1}{I_0(\beta L)}$$

$$\theta = \frac{I_o(\beta x)}{I_0(\beta L)}$$

Thus, we get the temperature profile:

$$T = T_\infty + \frac{(T_b - T_\infty)}{I_0(\beta L)} I_0(\beta x);\ \beta = \frac{P_L h}{A_L k}$$

We can determine *heat loss* by performing derivatives with respect to x of temperature profile

$$Q_{loss} = A(-k)\left.\frac{dT}{dx}\right|_{x=L}$$

$$\left.\frac{dT}{dx}\right|_{x=L} = \frac{(T_b - T_\infty)}{I_0(\beta L)} \beta I_1(\beta x)|_{x=L}$$

$$Q_{loss} = A(-k)\left.\frac{dT}{dx}\right|_{x=L} = \frac{(T_b - T_\infty)}{I_0(\beta L)} \beta I_1(\beta x)|_{x=L}(-k)(A_L)$$

$$Q_{loss} = \frac{(T_b - T_\infty)}{I_0(\beta L)} \beta(-k)A_L I_1(\beta L)$$

$$Q_{ideal} = (T_b - T_\infty)h\left(\left(w\sqrt{\left(\frac{t}{2}\right)^2 + L^2}\right)2 + \left(\frac{1}{2}Lt2\right)\right)$$

Since t<<w, we could neglect the term t ≈0.

$$Q_{loss} = (T_b - T_\infty)hWL2$$

We can determine *fin efficiency* (η_f) which is defined by

$$\eta_f = \frac{actual\ rate\ of\ heat\ loss\ from\ the\ fin}{rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_b}$$

$$\eta_f = (T_b - T_\infty)\beta(+k)A_L \frac{I_1(\beta L)}{I_0(\beta L)} \frac{1}{(T_b - T_\infty)hWL2}$$

Remark: $\beta = \sqrt{\frac{2}{t}\frac{h}{k}}$;

$$\eta_f = \sqrt{\frac{2}{t}\frac{h}{k}}\frac{(kWt)}{hWL2}\frac{I_1(\beta L)}{I_0(\beta L)}$$

$$\eta_f = \sqrt{\frac{2}{t}\frac{hk^2t}{kh^2 2^2}}\frac{1}{L}\frac{I_1(\beta L)}{I_0(\beta L)}$$

$$\eta_f = \frac{I_1(\beta L)}{\beta I_0(\beta L)}\sqrt{\frac{kt}{2h}}$$

We can determine *fin effectiveness* (ε_f)

$$Q_{no\ fin} = h(T_b - T_\infty)A_L$$

$$\varepsilon_f = \frac{Q_{actual}}{Q_{no\ fin}} = \frac{\frac{(T_b - T_\infty)}{I_0(\beta L)}\beta(-k)A_L I_1(\beta L)}{h(T_b - T_\infty)A_L}$$

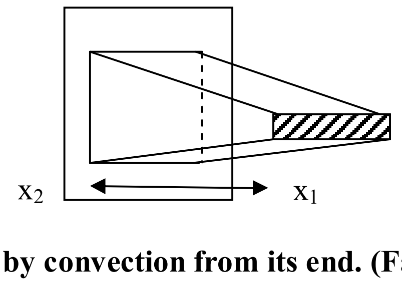
$$\varepsilon_f = \sqrt{\frac{2}{t}\frac{h}{k}}\frac{k}{h}\frac{I_1(\beta L)}{I_0(\beta L)}$$

$$\varepsilon_f = \sqrt{\frac{2}{t}\frac{k}{h}}\frac{I_1(\beta L)}{I_0(\beta L)}$$

<u>Case II</u>: The end of fin is insulated so that $\frac{dt}{dx} = 0$ at x=L (No heat is lost from the end or from the edges.)

BC1: $x = x_1$: $\left.\frac{dT}{dx}\right|_{x=x_1}$; $\left.\frac{d\theta}{dx}\right|_{x=x_1} = 0$

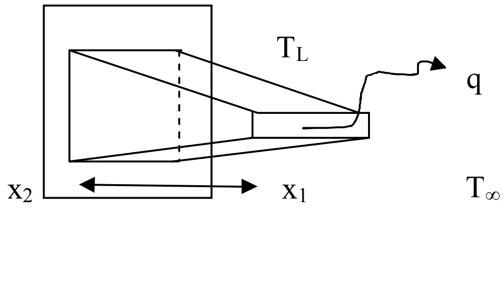
BC2: $x = x_2$: $T = T_b$; $\theta = 1$



<u>Case III</u>: The fin is of finite length and loses heat by convection from its end. (Facing convection BC at the tip of fin)

BC1: $q|_{x=x_1} = -\left.\frac{kdT}{dx}\right|_{x=x_1} = h(T_L - T_\infty)$

$$-(T_b - T_\infty)\left.\frac{d\theta}{dx}\right|_{x=x_1} = h(T_L - T_\infty)$$



$$-k\frac{d\theta}{dx}\bigg|_{x=x_1} = \frac{h(T|_{x=x_1} - T_\infty)}{T_b - T_\infty}$$

$$\frac{d\theta}{dx}\bigg|_{x=x_1} = \frac{-h\emptyset|_{x=x_1}}{k}$$

BC2: $x = L: T = T_b;\ \theta = 1$