Example 16 Heat conduction in a cooling Fin

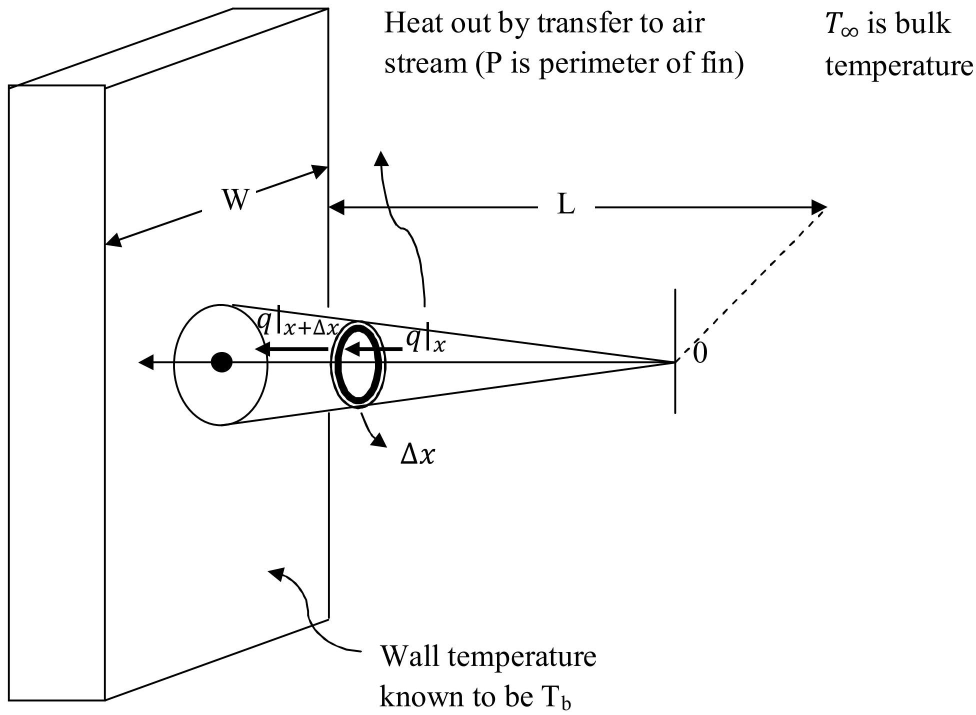


Figure 16 A simple cooling fin

Assume:

$$\frac{x}{L} = \frac{A_x}{A_L} = \frac{P_x}{P_L}$$

$$A_x = \frac{A_L}{L}x; P_x = \frac{P_L}{L}x$$

Conduction heat balance around shell at steady state

[Rate of heat in] – [Rate of heat out] - [Rate of heat loss] = 0

$$q|_x \cdot A_x - q|_{x+\Delta x} \cdot A_x - h(T - T_\infty)P_x\Delta x = 0$$

Substitute $A_x = \frac{A_L}{L}x$ and $P_x = \frac{P_L}{L}x$ into the above equation

$$q|_x \cdot \frac{A_L}{L}x - q|_{x+\Delta x} \cdot \frac{A_L}{L}x - h(T - T_\infty)\frac{P_L}{L}x\Delta x = 0$$

Divided by $\frac{A_L}{L}\Delta x$ through equation and taking limit $\Delta x \to 0$ we get

$$-\lim_{\Delta x \to 0}\left(\frac{q|_{x+\Delta x} \cdot x - q|_x \cdot x}{\Delta x}\right) - h(T - T_\infty)\frac{P_L}{A_L}x = 0$$

From the definition of derivative equation we obtain

$$-\frac{\mathrm{d}(x \cdot q_x)}{\mathrm{dx}} - h(T - T_\infty)\frac{P_L}{A_L}x = 0$$

$$-\frac{\mathrm{d}(x \cdot q_x)}{\mathrm{dx}} = h(T - T_\infty)\frac{P_L}{A_L}x$$

Apply Fourier's law;

$$\frac{\mathrm{k}}{\mathrm{x}}\frac{\mathrm{d}}{\mathrm{dx}}\left(x\frac{dT}{dx}\right) = h(T - T_\infty)\frac{P_L}{A_L}$$

$$\frac{1}{x}\frac{d}{dx}\frac{xdT}{dx} = (T - T_\infty)\frac{P_L h}{kA_L} \qquad (2)$$

Set dimensionless group: $\theta = \frac{(T - T_\infty)}{(T_b - T_\infty)}$: $dT = (T_b - T_\infty)d\theta$, equation (2) becomes:

$$d\theta = d\left[\frac{(T - T_\infty)}{(T_b - T_\infty)}\right]$$

$$d\theta = \frac{1}{(T_b - T_\infty)}d[(T - T_\infty)]$$

$$d\theta = \frac{1}{(T_b - T_\infty)}[dT + 0]$$

$$(T_b - T_\infty)d\theta = dT$$

$$\frac{1}{x}\frac{d}{dx}\left(\frac{xd\theta}{dx}\right) = \frac{(T - T_\infty)}{(T_b - T_\infty)}\frac{P_L h}{A_L k}$$

$$\frac{1}{x}\frac{d}{dx}\left(\frac{xd\theta}{dx}\right) = \theta\frac{P_L h}{A_L k}$$

$$\frac{d}{dx}\left(x \cdot \frac{d\theta}{dx}\right) = \theta x\frac{P_L h}{A_L k}$$

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - \theta x\left(\frac{P_L h}{A_L k}\right) = 0$$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - x^2\theta\left(\frac{P_L h}{A_L k}\right) = 0$$

Set: $\frac{P_L h}{A_L k} = \beta^2$;

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - x^2\beta^2\theta = 0$$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - (x^2\beta^2 - 0^2)\theta = 0 \qquad (3)$$

From Modified Bessel:

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - (x^2\beta^2 + n^2)\theta = 0$$

Solution

$$\theta = AI_0(\beta x) + BK_0(\beta x) \qquad (4)$$

<u>Case I</u>: The fin is very long, and the temperature at the end of the fin is essentially that of the surrounding fluid.

Assume long fin; $L \rightarrow \infty$

BC1: $x = 0; \left.\frac{dT}{dx}\right|_{x=0} = 0: \left.\frac{d\theta}{dx}\right|_{x=0} = 0$

BC2: $x = L; T = T_b: \theta = 1$

$$\frac{d\theta}{dx} = A\frac{dI_0(\beta x)}{dx} + \frac{BdK_0(\beta x)}{dx}$$

$$\frac{d\theta}{dx} = A\beta I_1(\beta x) - BK_1\beta(\beta x)$$

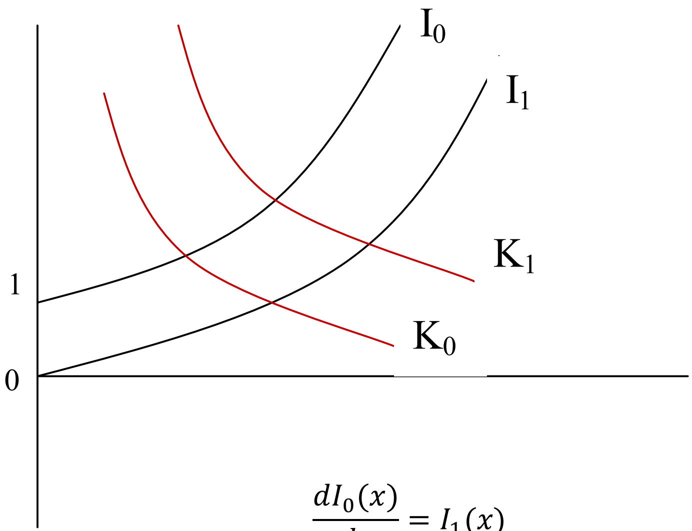
Use BC1 to substitute into eqn (3);

$$0 = A\underbrace{I_1(0)}_{0} - B\underbrace{K_1(0)}_{\infty}$$

Finite = 0 ∞

Because term $K_1(0) = \infty$ and the equation is finite, we have to force B = 0 to valid the equation. Eqn (4) becomes:

$$\theta = AI_0(\beta x) \qquad (5)$$



$$\frac{dI_0(x)}{dx} = I_1(x)$$

$$\frac{dK_0(x)}{dx} = -k_1(x)$$

Use BC2 substitute into equation (5)

$$1 = AI_0(\beta L)$$

$$A = \frac{1}{I_0(\beta L)}$$

$$\theta = \frac{I_o(\beta x)}{I_0(\beta L)}$$

Thus, we get the temperature profile:

$$\boxed{T = T_\infty + (T_b - T_\infty)\frac{I_0(\beta x)}{I_0(\beta L)};\ \beta = \frac{P_L h}{A_L k}}$$

We can determine *heat loss* by performing derivatives with respect to x of temperature profile

$$Q_{loss} = A(-k)\left.\frac{dT}{dx}\right|_{x=L}$$

$$\left.\frac{dT}{dx}\right|_{x=L} = \frac{(T_b - T_\infty)}{I_0(\beta L)}\beta I_1(\beta x)|_{x=L}$$

$$Q_{loss} = A(-k)\left.\frac{dT}{dx}\right|_{x=L} = \beta(T_b - T_\infty)\frac{I_1(\beta x)|_{x=L}}{I_0(\beta L)}(-k)(A_L)$$

$$Q_{loss} = -kA_L(T_b - T_\infty)\beta\frac{I_1(\beta L)}{I_0(\beta L)}$$

$$Q_{ideal} = h(T_b - T_\infty)\pi R\sqrt{R^2 + L^2}$$

$$Q_{no\ fin} = (T_b - T_\infty)\pi R_b^2$$

We can determine *fin efficiency* (η_f) which is defined by

$$\eta_f = \frac{actual\ rate\ of\ heat\ loss\ from\ the\ fin}{rate\ of\ heat\ loss\ from\ an\ isothermal\ fin\ at\ T_w}$$

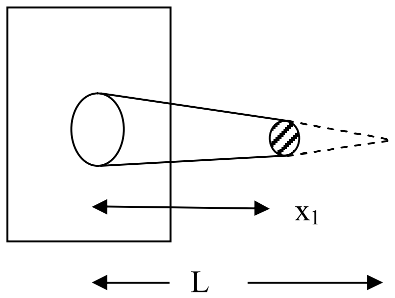
$$\eta_f = \frac{Q_{loss}}{Q_{ideal}}$$

We can determine *fin effectiveness* (ε_f)

$$\varepsilon_f = \frac{Q_{actual}}{Q_{no\ fin}}$$

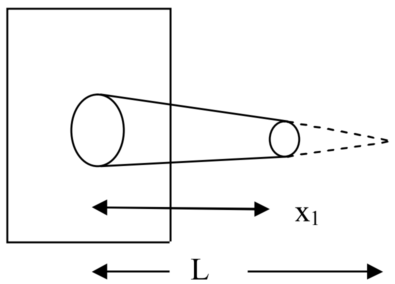
<u>Case II</u>: The end of fin is insulated so that $\frac{dt}{dx} = 0$ **at** x=L **(No heat is lost from the end or from the edges.)**

BC1: $x = x_1$: $\left.\frac{dT}{dx}\right|_{x=x_1}$; $\left.\frac{d\theta}{dx}\right|_{x=x_1} = 0$



BC2: $x = L$: $T = T_b$; $\theta = 1$

<u>Case III</u>: The fin is of finite length and loses heat by convection from its end. (Facing convection BC at the tip of fin)



BC1: $q|_{x=x_1} = -\left.\frac{kdT}{dx}\right|_{x=x_1} = h(T_L - T_\infty)$

$-(T_b - T_\infty)\left.\frac{d\theta}{dx}\right|_{x=x_1} = h(T_L - T_\infty)$

$-k\left.\frac{d\theta}{dx}\right|_{x=x_1} = \frac{h(T|_{x=x_1} - T_\infty)}{T_b - T_\infty}$

$\left.\frac{d\theta}{dx}\right|_{x=x_1} = \frac{-h\emptyset|_{x=x_1}}{k}$

BC2: $x = L$: $T = T_b$; $\theta = 1$