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Physica D 237 (2008) 2482–2486
PHYSICA D
www.elsevier.com/locate/physd

An approach for characterizing coupling in dynamical systems

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Received 25 July 2007; received in revised form 21 January 2008; accepted 6 March 2008
Available online 16 March 2008

Communicated by R. Roy

Abstract

The study of coupling in dynamical systems dates back to Christian Hyugens who, in 1665, discovered that pendulum clocks with the same length pendulum synchronize when they are near to each other. In that case the observed synchronous motion was out of phase. In this paper we propose a new approach for measuring the degree of coupling and synchronization of a dynamical system consisting of interacting subsystems. The measure is based on quantifying the active degrees of freedom (e.g. correlation dimension) of the coupled system and the constituent subsystems. The time-delay embedding scheme is extended to coupled systems and used for attractor reconstruction of the coupled dynamical system. We use the coupled Lorenz, Rossler and Hénon model systems with a coupling strength variable for evaluation of the proposed approach. Results show that we can measure the active degrees of freedom of the coupled dynamical systems and can quantify and distinguish the degree of synchronization or coupling in each of the dynamical systems studied. Furthermore, using this approach the direction of coupling can be determined.

Keywords: Coupling; Chaotic systems; Nonlinear dynamics; Time-delay embedding; State space

1. Introduction

Coupling and synchronization are common phenomena that occur in nature, e.g. in biological and physiological systems, as well as in engineered systems. The study of coupling dates back at least to 1665 when Hyugens observed coupling in nearby pendulum clocks. Recent studies of synchronization have included periodic dynamical systems as well as coupled chaotic systems with a focus on a variety of different types of synchronization. Different notions of synchronization include: complete synchronization where the trajectories of two identical systems remaining exactly in step [1–3]; phase synchronization where the phase difference is asymptotically bounded [4,5]; generalized synchronization where the output of one system is functionally related of the output of a different system [6,7]; lag synchronization where the difference between the output of one system and the time-delayed output of a second system are asymptotically bounded [8]; almost synchronization where the differences between a subset of variables from one system and the corresponding subset of variables from a second system are asymptotically bounded [9].

In spite of these efforts, a unifying framework for the study of synchronization of coupled dynamical systems was not available. Brown and Kocarev [10] proposed a general definition of synchronization based on the idea that different types of synchronization could be captured in a single formalism. The system is decomposed into two subsystems, and the trajectories of each subsystem are mapped through functions g_1 and g_2, respectively. Then the two subsystems are "synchronized" if there exists a function h such that $\|h(g_1, g_2)\|$ is equal to zero or approaches zero asymptotically as t tends to infinity. The type of synchronization is then determined by the functions g_1, g_2 and h. An alternative, and simpler, definition is proposed in [11] based on the concept of a "synchronization function". A technique to visualize coupling is introduced in [12] along with a robust measure that quantifies coupling strength.

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doi:10.1016/j.physd.2008.03.003

The notion of synchronization has been recently extended to the case of interacting chaotic oscillators [13]. Several methods derived from different principles and concepts have been proposed for characterizing coupled dynamical systems including measuring degrees of synchronization and specifying the direction of coupling. Six different measures of synchronization including cross correlation, mutual information, Hilbert phase circular variance, wavelet phase circular variance, nonlinear interdependence and event synchronization, are reviewed in [13]. Interested readers should refer to [13] and the references therein.

In this paper, we present an approach for measuring a degree of synchronization in a coupled dynamical system that consists of interacting subsystems. Our approach is based on computation of the correlation dimension [14]. The key idea of the proposed approach, which we refer to as the dynamic complexity coherence measure, is to study coupling in a dynamical system by computing the correlation dimension of the coupled system and its subsystems. This information is then used to define a relative index of the active degrees of freedom of the coupled system and its subsystems to quantify the coupling. The approach is based on a new method for attractor reconstruction, called the concatenated embedding vector reconstruction method, that is used to provide information for assessing the coupling in a dynamical system. Further, information about the active degrees of freedom of the subsystems can lead to an inference regarding the direction of the coupling between the subsystems. The performance of the concatenated embedding vector reconstruction method and the dynamic complexity coherence measure are evaluated using the coupled Lorenz, Rossler and Hénon systems. The computational results show that the correlation dimension computed using the concatenated embedding vector reconstruction method for the attractor provides a good approximation to the active degrees of freedom of the coupled dynamical system. The dynamic complexity coherence measure exhibits good performance in quantifying and distinguishing the degrees of synchronization with a high degree of monotonicity in terms of coupling strength. We also compare the results of the dynamic complexity coherence measure to the six measures of synchronization studied in [13]. In addition, we show that from our results the direction of coupling can also be identified.

2. Background

2.1. State space and reconstruction

The state of a dynamical system is a point in a vector space, referred to as the state space, and the trajectory of a dynamical system is a path (collection of points) in the state space. In cases of general interest, a dynamical system can be defined by a set of ordinary differential equations of the general form: $\mathrm{d}x/\mathrm{d}t = f(x)$, $x \in \mathcal{R}^D$ where D characterizes the dimension or degrees of freedom of the dynamical system. Typically, the response of a dynamical system that is observed is not a trajectory in the state space but rather a scalar sequence of discrete (sampled) measurements. Thus we need to characterize the relationship of such observables to points in the state space in order to obtain more comprehensive information on the dynamics of the system.

Let $s[n]$ denote a scalar sequence (time series) of observables of a dynamical system that depends on the current state of the system taken at uniform sampling times, where $n = 0, 1, \ldots, N-1$ and N is the length of the time series s. The representation of the time series s in the state space is commonly accomplished by unfolding the time series into a higher dimensional space using the time-delay embedding scheme, often referred to as Takens' reconstruction [15]. The m-dimensional embedding vector $\mathbf{s} \in \mathcal{R}^m$ generated from the time series s is given by

$$\mathbf{s}_n = \left(s[n] \quad s[n+\tau] \quad \cdots \quad s[n+(m-1)\tau]\right)^{\mathrm{T}} \tag{1}$$

where $n = 0, 1, \ldots, N_e - 1$ and $N_e = N - (m-1)\tau$. Here m and τ are the embedding parameters and denote the embedding dimension and the time delay, respectively, and $\cdot^{\mathrm{T}}$ denotes vector transpose.

2.2. Correlation integral and dimension

In [14], the correlation integral and the correlation dimension were introduced for measuring the strangeness of an attractor, and for quantifying the active degrees of freedom of the dynamical system and characterizing the power-law behaviour of the system at the appropriate scale, respectively. The correlation integral of the time series s is defined by

$$C(r) = \lim_{N_c \to \infty} \sum_{i=0}^{N_e-1} \sum_{j=i+1}^{N_e-1} \frac{2}{N_c} \Theta\left(r - \|\mathbf{s}_i - \mathbf{s}_j\|\right) \tag{2}$$

where $N_c = N_e(N_e - 1)$ and the Heaviside function $\Theta(n) = 1$ if $n \geq 0$; 0, otherwise. The correlation integral $C(r)$ behaves as a power-law with exponent ν for small r, i.e. $C(r) \propto r^\nu$ where the exponent ν is defined as the correlation dimension. The correlation dimension can be used to characterize the complexity of a dynamical system on the attractor.

3. Methods

Here, a coupled dynamical system Φ consists of two subsystems X and Y defined by

$$\dot{\mathbf{x}} = \mathbf{f}_x(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \dot{\mathbf{y}} = \mathbf{f}_y(\mathbf{x}, \mathbf{y}). \tag{3}$$

Note: The method can also be applied to other models involving more than two interconnected subsystems.

3.1. Concatenated embedding vector reconstruction

The proposed reconstruction method, referred to as concatenated embedding vector reconstruction, is used to characterize the state information of coupled dynamical systems. The embedding vector of Φ using the concatenated embedding vector reconstruction, $\tilde{\mathbf{s}}_n$, is given by

$$\tilde{\mathbf{s}}_n = \left((\mathbf{s}_n^x)^{\mathrm{T}} \quad (\mathbf{s}_n^y)^{\mathrm{T}}\right)^{\mathrm{T}} \tag{4}$$

where $\mathbf{s}_n^x$ and $\mathbf{s}_n^y$ denote, respectively, the embedding vectors of X and Y using the time-delay reconstruction method given in Eq. (1), with the corresponding embedding dimensions m^x and m^y, and time delays τ^x and τ^y. The correlation dimension computed using the embedding vector $\tilde{\mathbf{s}}_n$ specifies the active degrees of freedom or complexity of Φ on the attractor.

3.2. Dynamic complexity coherence measure

The active degrees of freedom of the coupled dynamical system Φ depend on the interdependency of the subsystems X and Y. As such, the dynamic behaviour of Φ is influenced by the dynamics of both X and Y. If, for example, the subsystems X and Y are independent we would expect that the active degrees of freedom of Φ are equal to the sum of the active degrees of freedom of X and Y. On the other hand, if X and Y are coupled, we expect the active degrees of freedom of Φ to be reduced and less than in the case when the subsystems X and Y are independent. Accordingly, we measure the degree of synchronization between the subsystems in Φ as the ratio of the sum of active degrees of freedom of the subsystems X and Y to the active degrees of freedom of the total system Φ.

The dynamic complexity coherence measure for the coupled dynamical system Φ is given by

$$\Gamma_\Phi = \frac{\nu^x + \nu^y}{\nu^\Phi} \tag{5}$$

where ν^Φ, ν^x, and ν^y denote the correlation dimensions of Φ, X, and Y, respectively. The dynamic complexity coherence measure Γ_Φ increases with the degree of synchronization of X and Y.

Also, for unidirectional coupling in the dynamical system Φ: for example, if X is independent of Y but Y is dependent on X, the variability in the correlation dimensions of X and Y can be used to specify the direction of the coupling. Because the subsystem Y is driven by the subsystem X, the dynamics of Y are dependent on the dynamics of X as well as on the type and strength of the coupling. Therefore, the variability in the correlation dimension for epochs of time series data for subsystem Y will be greater than that for subsystem X.

4. Results

4.1. Computational experiments of model systems

Three coupled dynamical model systems that were recently studied in [13] and have also been used in several previous studies are used in the computational experiments. These model systems include the coupled Lorenz, Rossler and Hénon systems. Each coupled dynamical model system includes unidirectional coupling of either two Lorenz, two Rossler or two Hénon systems, with subsytems X and Y. The coupled Lorenz system is given by

$$\begin{aligned}
\dot{x}_1 &= 10(x_2 - x_1)\\
\dot{x}_2 &= x_1(28 - x_3) - x_2\\
\dot{x}_3 &= x_1 x_2 - \frac{8}{3}x_3\\
\dot{y}_1 &= 10(y_2 - y_1)\\
\dot{y}_2 &= y_1(28.001 - y_3) - y_2\\
\dot{y}_3 &= y_1 y_2 - \frac{8}{3}y_3 + C(x_3 - y_3).
\end{aligned} \tag{6}$$

The coupled Rossler system is given by

$$\begin{aligned}
\dot{x}_1 &= -0.95x_2 - x_3\\
\dot{x}_2 &= 0.95x_1 + 0.15x_2\\
\dot{x}_3 &= 0.2 + x_3(x_1 - 10)\\
\dot{y}_1 &= -1.05y_2 - y_3 + C(x_1 - y_1)\\
\dot{y}_2 &= 1.05y_1 + 0.15y_2\\
\dot{y}_3 &= 0.2 + y_3(y_1 - 10).
\end{aligned} \tag{7}$$

The coupled Hénon system is given by

$$\begin{aligned}
x_1[n+1] &= 1.4 - x_1^2[n] + 0.3x_2[n]\\
x_2[n+1] &= x_1[n]\\
y_1[n+1] &= 1.4 - (Cx_1[n]y_1[n] + (1-C)y_1^2[n])\\
&\quad + 0.3y_2[n]\\
y_2[n+1] &= y_1[n].
\end{aligned} \tag{8}$$

The degree of synchronization in these systems is controlled by the coupling strength parameter C. In the experiments, the coupling strength C for the coupled Lorenz system and the coupled Rossler system is varied from 0 to 2 in steps of 0.1, while the coupling strength C for the coupled Hénon system is varied from 0 to 0.8 in steps of 0.04. The coupled Lorenz and Rossler systems are generated using the 4th order Runge-Kutta integration method with integration step sizes of 0.01 and 0.10, respectively.

The computations associated with the coupled Lorenz systems use the x_1 and y_1 time series, while the computations associated with the coupled Rossler and Hénon systems use the x_2 and y_2 time series. The time-delay embedding parameters for the coupled Lorenz, Rossler, and Hénon systems are $m = 8$ and $\tau = 12$, $m = 8$ and $\tau = 6$, and $m = 6$ and $\tau = 1$, respectively. For comparison, we compute the correlation dimensions of the coupled Lorenz, Rossler, and Hénon systems using the actual vectors $\mathbf{x}$ and $\mathbf{y}$ without the need for time-delay embedding in reconstructing the attractor. For each experiment, the correlation dimensions of 50 consecutive epochs, each 8000 samples in length, are computed. Further, the method for measuring the degree of monotonicity presented in [13] is used to determine how well we are able to distinguish between different degrees of coupling in these model systems.

4.2. Active degrees of freedom

The average correlation dimensions of the coupled Lorenz, Rossler and Hénon systems computed using the concatenated embedding vectors for reconstruction are shown in Fig. 1. As

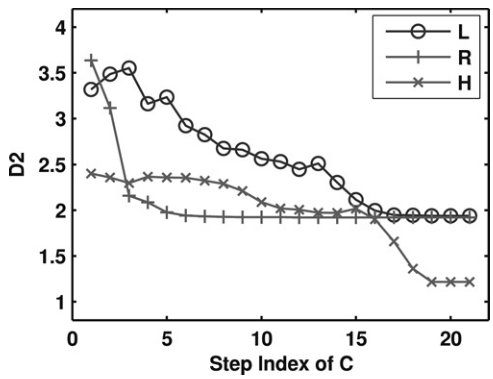

Fig. 1. The average correlation dimensions of the coupled Lorenz, Rossler and Hénon systems computed using the concatenated embedding state vector reconstruction.

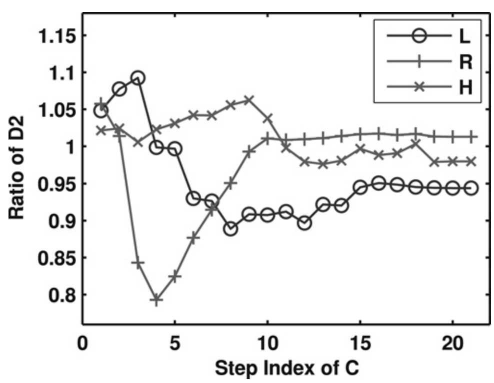

Fig. 2. The average ratios of the correlation dimensions of the coupled Lorenz, Rossler and Hénon systems computed using the concatenated embedding state vector reconstruction and computed from the actual state vectors **x** and **y**.

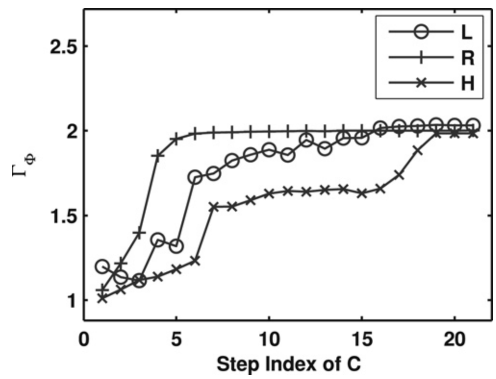

Fig. 3. The average dynamics complexity coherence measures of the coupled Lorenz, Rossler and Hénon systems.

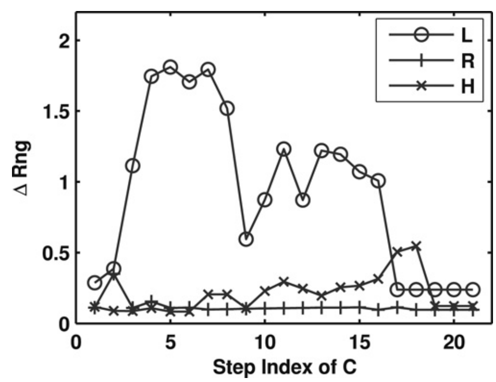

Fig. 4. The differences between ranges of the correlation dimensions of the sub-dynamical systems Y and X of the coupled Lorenz, Rossler and Hénon systems.

expected, the correlation dimensions of the coupled Lorenz, Rossler and Hénon systems tend to decrease as the coupling strength C increases. Fig. 2 illustrates the average ratios of the correlation dimensions computed using the concatenated embedding vectors for reconstruction and these are compared to those computed from the actual vectors **x** and **y**. The coupled Hénon system has the smallest difference between the correlation dimensions computed using the concatenated embedding vectors and the actual state vectors. The average ratios of the correlation dimensions of the coupled Lorenz, Rossler and Hénon systems are, respectively, 0.955, 0.973, and 1.009. In general, larger differences between the correlation dimensions occur at weaker coupling strengths.

4.3. Degrees of synchronization

Fig. 3 illustrates the average values of the dynamic complexity coherence measure Γ_Φ for the coupled Lorenz, Rossler and Hénon systems. Obviously, the dynamic complexity coherence measure tends to increase as the coupling strength C increases. The increasing trend of the dynamic complexity coherence measure is somewhat monotonic with coupling strength and the degrees of monotonicity [13] for the coupled Lorenz, Rossler and Hénon systems are, respectively, 0.914, 0.914, 0.929. The coupled Lorenz, Rossler and Hénon systems begin to be fully coupled, that is Γ_Φ does not continue to increase with increased coupling strength, at about $C = 1.60, 0.80, 0.72$, respectively.

4.4. Direction of coupling

The differences between the ranges of the correlation dimensions for the Y and X subsystems of the coupled Lorenz, Rossler and Hénon systems are shown in Fig. 4. The ranges for the correlation dimensions of the Y subsystem are in general larger than those of the X subsystem, thereby indicating that X has more of a direct influence on Y than Y has on X.

5. Discussion

In this paper we have introduced concatenated embedding vector reconstruction for characterizing information about coupling in interconnected dynamical systems. The concatenated embedding vector is then used to determine the active degrees of freedom of the overall system in terms of one-dimensional time series observations from the two subsystems as quantified by the correlation dimension. A measure of coupling, referred to as the dynamic complexity coherence measure and defined as the ratio of the sum of the correlation dimensions (active degrees of freedom) of the subsystems to the correlation dimension (active degrees of freedom) of the coupled dynamical system, is then used to quantify the degree of synchronization in the dynamical system. We propose that the direction of the strongest coupling between the subsystems X and Y can be investigated by examining the range of variability of their correlation dimension estimates.

From the computational results given for model systems, the correlation dimension computed using the concatenated embedding vector reconstruction is a good approximation of the active degrees of freedom of the coupled dynamical system as quantified by the computation of the correlation dimension from the actual state vector of the system. Further, the dynamic complexity coherence measure can be used to quantify and distinguish the coupling strength C of coupled dynamical systems with a high degree of monotonicity. Using the mean values of the degrees of monotonicity as a measure of the effectiveness of a given coupling measure, the dynamic complexity coherence measure has better performance in quantifying and distinguishing the degrees of synchronization of the coupled Lorenz system than all six measures evaluated in [13]. In addition, the dynamic complexity coherence measure when applied to the coupled Hénon system performs better than the average of all six measures reported in [13], while the dynamic complexity coherence measure applied to the coupled Rossler system performs better than only one of these six measures. Because the dynamic complexity coherence measure is based on the correlation dimension, its primary drawback is the computational complexity of the calculation of the correlation integral and dimension.

In subsequent work we will be investigating the application of the dynamic complexity coherence measure to model system data corrupted by noise as well as real data. The analysis of potentially noisy time series using the correlation dimension can be problematic as the noise tends to inflate the estimate of the correlation dimension. If precautions to minimize the impact of noise on the computation are not taken, the results can be particularly misleading. For this reason the use of surrogate data and statistical analysis of the computational results is important in reaching any conclusions about the system being studied. In the application to coupling, as described in this paper, the influence of noise may not be as dramatic as in an application where the estimate of the correlation dimension is not being used in a comparative analysis. It is reasonable to assume that in the analysis of a coupled system the noise uniformly influences all time series measurements and because the dynamic complexity coherence measure is a ratio, meaningful results may still be obtained. Of course, this remains to be tested and validated through further computational studies.

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