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จึงเรียนมาเพื่อโปรดพิจารณา

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Research Participation Agreement

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<td>1. Suchart Limkatanyu</td>
<td>40% - Writing the paper; Formulating the problem; Implementing the model; Interpreting and discussing the results.</td>
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<td>2. Paitoon Ponbunyanon</td>
<td>25% - Writing the paper; Partially preparing numerical examples; Discussing the simulation results.</td>
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<td>3. Woraphot Prachasaree</td>
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Correlation between beam on Winkler-Pasternak foundation and beam on elastic substrate medium with inclusion of microstructure and surface effects

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Abstract

A novel beam-elastic substrate element with inclusion of microstructure and surface energy effects is proposed in this paper. The modified couple stress theory is employed to account for the microstructure-dependent effect of the beam bulk material while Gurtin-Murdoch surface theory is used to capture the surface energy-dependent size effect. Interaction mechanism between the beam and the surrounding substrate medium is represented by the Winkler foundation model. The governing differential equilibrium and compatibility equations of the beam-elastic substrate system are consistently derived based on virtual displacement and virtual force principles, respectively. Both essential and natural boundary conditions of the system are also obtained. Two modified Tonti’s diagrams are presented to provide the big picture of both displacement-based and force-based formulations of the system. Due to similarity between the current problem and the one related to the beam on Winkler-Pasternak foundation, the so-called “natural” beam-Winkler-Pasternak foundation element coined by the authors is employed to perform two numerical simulations to study the characteristics and behaviors of a beam-substrate system with inclusion of microstructure and surface effects.

Keywords: Finite beam element; Winkler-Pasternak foundation; Modified couple stress theory; Surface elasticity theory; Displacement-based formulation; Force-based formulation; Virtual displacement principle; Virtual force principle

1. Introduction

Micro- and nano-sized beams have found a wide spectrum of applications in micro- and nano-scale mechanical devices and systems; such as micro and nano beams, microfilms, biosensors, atomic force microscopes, nanotubes, nanowires, micro/nano electro-mechanical systems (M/NEMS) [1-10]. To fully take advantage of these micro- and nano-sized beams on the development of powerful micro- and nano-scale mechanical devices and systems, profound understanding on their behavior and characterization at such very small scale is essential and imposes a significant challenge to the research community. Both experimental and analytical researches on beam responses at micro- and nano- scales have been studied comprehensively by many scientists and engineers. At micro and nano level, experimental work on such small beams is extremely difficult to be conducted and is prohibitively expensive due to necessity of high precision testing devices and unique testing procedures. Therefore, numerical simulation has been widely used in the research community as an alternative to characterize the structural response at micro- and nano-scales. Several numerical models with different degrees of complexity have been developed in literatures to study the micro and nano structural system. Generally, these numerical models can be categorized broadly into two groups: atomistic model and continuum-mechanics model [11]. The atomistic approach emphasizes on atomic modeling and embraces several techniques such as: classical molecular dynamics [12], tight binding molecular dynamics [13], and density functional theory [14]. Simulation performed with the atomistic approach provides comprehensive details but are hampered by high computational costs, thus only a system with small numbers of molecule and atom can be realistically investigated using this approach [15]. The continuum-mechanics approach serves as an attractive alternative to characterize the micro and nano structural responses and is applicable to model large-scale structural systems. However, size-dependent effect and small-scale effect inherent to micro- and nano-sized structures are not included in the classical continuum-mechanics theory. The size-dependent effect is a result from energy correspondent to the atoms at the free surface of micro- and nano-sized struc-
tures while the small-scale effect is caused by long-range inter-atomic interactions. To account for these two effects, surface elasticity theory and higher-order continuum-mechanics theory have been incorporated into the classical continuum-mechanics theory.

The surface elasticity theory was first proposed by Gurtin and Murdoch [16, 17] and has been employed to represent the size-dependent effect caused by surface stress and surface elasticity at micro- and nano-scales. As the size of a structure gets smaller, the surface free energy caused by the surface stress and surface elasticity could no longer be neglected in comparison with the bulk energy as that in the classical continuum-mechanics theory due to the high surface-to-volume ratio.

To include the material-length scale effect, several higher-order continuum-mechanics theories have been proposed in literatures. In the micropolar elasticity theory proposed in the early twentieth century by Cosserat and Cosserat [18], additional rotational degrees of freedom at each material point are appended to include the intrinsic length scale into the continuum body. As a special case of the Cosserat micropolar elasticity theory, classical couple stress theory was proposed by several researchers in the Sixties [19–22] which contains four material parameters (two classical and two nonclassical) for an elastic isotropic body. The modified strain gradient elasticity theory was proposed by Lam et al. [23], introducing a new equilibrium equation in addition to the classical equilibrium equations. Thus, this higher-order continuum-mechanics theory requires two classical and three nonclassical material parameters for an elastic isotropic body. Another widely employed higher-order continuum-mechanics theory is the nonlocal elasticity theory proposed by Eringen [24–26] and Eringen and Edelen [27]. The essence of this theory is in its assertion that the stress at a reference point depends on the strain not only at a particular point but also at all other points throughout the elastic body to account for the material-length scale effect. Two nonclassical material parameters are required in this higher-order continuum-mechanics theory besides two classical material parameters for an elastic isotropic body.

Several researchers have formulated various beam models with and without surface effect based on the aforementioned higher-order continuum-mechanics theories. For example, Anthoine [28] studied the pure bending behavior of a circular cylinder using the beam model based on the classical couple stress theory. Papargyri-Besikou et al. [29] developed a higher-order beam model using strain gradient elasticity theory and surface energy of Vardoulakis and Sulem [30]. Kong et al. [31] performed static and dynamic analysis of micro beams using the strain gradient elasticity theory. Alshorbagy et al. [7] formulated the finite beam element based on nonlocal elasticity theory and later Mahmoud et al. [32] enhanced this beam element by incorporating Gurtin-Murdoch surface effect into the element.

Considering the complexities in calibrating the material length-scale parameter [23, 33, 34] and the approximate nature inherent to the beam theory, the beam model with minimal material parameters is desirable from the practical point of view. The modified couple stress theory proposed by Yang et al. [35] makes such a beam model possible since only one material length-scale parameter is required. The very first beam model based on the modified couple stress theory was proposed by Park and Gao [4] using the Euler-Bernoulli beam theory. Later, Ma et al. [5] extended this beam model to include shear deformation using Timoshenko beam theory. Recently, Gao and Mahmoud [36] combined the modified couple stress theory with the Gurtin-Murdoch surface theory to formulate the Euler-Bernoulli beam model with inclusion of microstructure and surface effects.

In micro- and nano-scale mechanical devices and systems, beams are usually integrated into larger structures through substrate media. Therefore, interaction mechanism between the beam and the surrounding substrate media plays a crucial role in controlling the performance and characterizing the response of those systems. Several researchers have recently investigated the problem of beams resting on elastic substrate media. For example, Zhang and Zhao [37] developed a nanowire model lying on an adhesive receding contact foundation; Khajehansari et al. [38] performed parametric studies of silver nanowires resting on Winkler-Pasternak elastic substrate media using an analytical solution to the problem; Malekzadeh and Shojaei [39] investigated surface and nonlocal effects on nonlinear free vibration of nanowires supported by an elastic medium using both Euler-Bernoulli beam theory and Timoshenko beam theory.

In this research, the problem of beams on elastic substrate media with inclusion of microstructure and surface effects is of main interest due to its wide spectrum of applications in micro- and nano-scale mechanical devices and systems and the corresponding beam model is formulated using the virtual force principle. The general idea of the model formulation stems from the beam-foundation model developed by Limkatanyu et al. [40] and the beam model incorporating the microstructure and surface effects proposed by Gao and Mahmoud [36]. The modified couple stress theory [35] is used to account
for the microstructure-dependent effect of the bulk beam material while the Gurtin-Murdoch surface theory [16, 17] is employed to characterize the beam surface layer. The interaction between the beam and the surrounding substrate media is represented by the Winkler foundation [41].

Organization of this paper is as follows: The Euler-Bernoulli beam hypothesis, the modified couple stress theory, and the surface elasticity theory forming a set of basic ingredients for the model formulation are firstly described. Then, the governing differential equilibrium and compatibility equations of the problem are derived based on virtual displacement and virtual force principles, respectively. Both natural and essential boundary conditions of the problem are also obtained. The sectional force-deformation relations are subsequently established. Two modified Tonti's diagrams are presented to provide the big picture of both displacement-based and force-based formulations of the problem. Due to similarity between the current problem and the one related to the beam on Winkler-Pasternak foundation, the so-called "natural" element stiffness model formulated by Limkatanyu et al. [40] is finally employed to perform two numerical simulations to study the characteristics and behaviors of a beam-substrate system with inclusion of microstructure and surface effects. The first simulation involves investigation of the response of the beam resting on an elastic substrate. The second simulation examines the influences of several system parameters on contact stiffness and demonstrates the size-dependent effect on the system response.

2. Basic ingredients

2.1 Euler-Bernoulli beam kinematics

The kinematics description of a generic point $P$ on the Euler-Bernoulli beam cross-section is shown in Fig. 1. The cross-section $ab$ is normal to the longitudinal axis of the undeformed beam. In the deformed configuration, the deformed cross-section $a'b'$ remains plane and is normal to the longitudinal axis. This simply implies that the displacement at the point $P$ with a distance $y$ form the reference axis $x$ is:

$$u_x(x, y) = -y \frac{dv(x)}{dx}, u_y(x) = v(x) \text{; and } u_z(x) = 0$$

(1)

where $u_x(x, y)$, $u_y(x)$, and $u_z(x)$ are the displacement components of the point $P$ along the $x$, $y$, and $z$ axes, respectively; and $v(x)$ is the transverse displacement of the point on the reference axis $x$.

2.2 Modified couple stress theory

In this paper, the modified couple stress theory proposed by Yang et al. [35] is employed to describe the length-scale effect inherent to micro- and nano-sized structures. This theory stems from the classical couple stress theory proposed by several researchers in the Sixties [19-22]. The modified couple stress theory is in preference to the classical couple stress theory due to its requirement of only one additional material length-scale parameter and its inclusion of a symmetric couple stress tensor.

In the classical elasticity theory, the constitutive relation between the symmetric stress tensor $\sigma_j$ and the infinitesimal strain tensor $e_j$ reads [42]:

$$\sigma_j = \lambda \varepsilon_{kl} \delta_j + 2 \mu \varepsilon_j$$

(2)

where $\lambda$ and $\mu$ are Lame constants; $\delta_j$ is the Kronecker delta; and $e_j$ is defined as the symmetric part of the displacement gradient tensor $u_{ij}$:

$$e_j = \frac{1}{2} (u_{ij,j} + u_{ij,i})$$

(3)

In the modified couple stress theory, one additional constitutive relation between the deviatoric part of the couple stress tensor $m_j$ and the symmetric curvature tensor $\chi_j$ is supplied to account for the length-scale effect and is defined as:

$$m_j = 2 l^2 \mu \chi_j$$

(4)

where $l$ is the material length-scale parameter and $\chi_j$ is defined as the symmetric part of the rotation gradient tensor $\theta_j$:

$$\chi_j = \frac{1}{2} (\theta_{ij,j} + \theta_{ij,i})$$

(5)

with the rotation vector $\theta_j$ being defined as:

$$\theta_j = \frac{1}{2} e_{j\alpha} u_{\alpha,i}$$

(6)

where $e_{\alpha\beta}$ is the permutation symbol.

Following the sectional kinematics of Eq. (1), the non-zero components of the strain tensor $e_j$ and the symmetric curvature tensor $\chi_j$ are expressed in terms of the beam transverse displacement $v(x)$ as:

$$e_{x\alpha}(x, y) = -y \frac{d^2 v(x)}{dx^2}, \chi_{x\alpha}(x) = \frac{1}{2} \frac{d^2 v(x)}{dx^2} \text{.}$$

(7)

Substituting Eq. (7) into the constitutive relations of Eqs. (2) and (4), the non-zero components of the stress tensor $\sigma_j$ and the deviatoric part of the couple stress tensor $m_j$ are expressed in terms of the beam transverse displacement $v(x)$ as:

$$\sigma_{x\alpha}(x, y) = \frac{y E (1 - v)}{(1 + v)(1 - 2v)} \frac{d^2 v(x)}{dx^2} \text{.}$$
\[ \sigma_{yx}(x, y) = \sigma_{xz}(x, y) = \frac{v \sigma_{yy}(x, y)}{(1 - \nu^2)(1 - 2\nu)} \]
\[ m_{yz}(x) = m_{zx}(x) = \frac{E \mu}{2(1 + \nu)} \left( \frac{d^2 v(x)}{dx^2} \right) \tag{8} \]

where \( E \) and \( \nu \) are the Young’s modulus and the Poisson’s ratio, respectively, and are expressed in terms of Lame constants \( \lambda \) and \( \mu \) as:
\[ \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)}. \tag{9} \]

### 2.3 Surface elasticity theory

To account for surface effect on the micro- and nano-sized structures, the Gurtin-Murdoch continuum model [16, 17] is employed. In this nonclassical continuum model, the beam cross-section is considered to be consisted of a solid core and an outer surface shell perfectly bonded to its core, as shown in Fig. 2. Following the surface theory proposed by Cammarata [43], the outer surface is considered a mathematically-zero thickness layer with a distinct elastic modulus from its core material. The constitutive relations of the surface proposed by Gurtin and Murdoch [16, 17] are:
\[ \tau_{ss} - \tau_0 = (\lambda_0 + 2\mu_0) e_{ss}^s \quad \text{and} \quad \tau_{ss} = \tau_0 e_{ss}^e \tag{10} \]

where \( \tau_0 \) is the residual surface stress under unconstrained conditions; \( \lambda_0 \) and \( \mu_0 \) are surface Lame constants and can be obtained via atomistic simulations [44]; \( \tau_{ss} \) and \( \tau_{es} \) are non-zero membrane stresses in the elastic surface; and \( e_{ss}^s \) and \( e_{ss}^e \) are elastic surface deformations and are defined as [36]:
\[ e_{ss}^s(x, y) = \frac{du_s(x, y)}{dx} \quad \text{and} \quad e_{ss}^e(x, y) = n_y \frac{du_s(x)}{dx} \tag{11} \]

with \( n_y \) being the y-component of the unit outward normal vector to the beam-section lateral surface.

Substituting Eq. (1) into Eq. (11) and then into Eq. (10), the non-zero membrane stresses in the elastic surface \( \tau_{ss} \) and \( \tau_{es} \) can be expressed in terms of the beam transverse displacement \( v(x) \) as:

![Fig. 2. An arbitrary beam cross-section: beam bulk and surface layer.](image)

\[ \tau_{ss}(x, y) - \tau_0 = -y (\lambda_0 + 2\mu_0) \frac{d^2 v(x)}{dx^2}. \tag{12} \]

### 3. Governing equations of beams on elastic substrate media with inclusion of microstructure and surface effects

#### 3.1 Differential equilibrium equation: the virtual displacement approach

The virtual displacement principle is employed to consistently derive the governing differential equilibrium equations and natural boundary conditions of a beam resting on elastic substrate medium with inclusion of microstructure and surface effects as shown in Fig. 3. Interaction between the beam and the surrounding substrate media is represented by the Winkler foundation [41]. The general form of the virtual displacement principle is written as:
\[ \delta W = \delta W_{int} + \delta W_{ext} = 0 \tag{13} \]

where \( \delta W \) is the system total virtual work; \( \delta W_{int} \) is the system internal virtual work; and \( \delta W_{ext} \) is the system external virtual work.

In the case of the beam on elastic substrate medium with inclusion of microstructure and surface effects, \( \delta W_{int} \) and \( \delta W_{ext} \) can be expressed as:
\[ \delta W_{int} = \int L \left[ \sum_{\sigma} (\sigma_{ss} + m_{ss}) \delta \varepsilon_{ss}(x, y) + m_{es} \delta \varepsilon_{es}(x) \right] dx \]
\[ + \int L \left[ \int_{A} \sum_{\sigma} \left( \sigma_{ss} \delta \varepsilon_{ss}(x, y) + 2m_{ss} \delta \varepsilon_{es}(x) + m_{es} \delta \varepsilon_{es}(x) \right) dA \right] dx \]

\[ \delta W_{ext} = -\int L \left( p_s(x) \delta v(x) - U^T \mathbf{P} \right) dx \tag{14} \]

\[ \delta W_{ext} = -\int L \left( p_s(x) \delta v(x) - U^T \mathbf{P} \right) dx \tag{15} \]
where $D_s(x)$ is the elastic-substrate force; $A_s(x)$ is the elastic-substrate deformation and is equal to the beam’s transverse displacement $v(x)$ following the Winkler foundation hypothesis [41]; $P_s(x)$ is the transverse distributed load; the vector $\mathbf{P} = \{P_1, P_2, P_3, P_4\}$ contains shear forces and moments acting at element ends; and the vector $\mathbf{U} = \{U_1, U_2, U_3, U_4\}^T$ contains their conjugate-work displacements and rotations. It is noted that the first, second, and third terms in Eq. (14) represent the contributions of the underlying elastic substrate, the bulk material, and the surface layer to the system’s internal virtual work, respectively.

Using Eqs. (1), (7) and (11), Eq. (14) can be expressed as:

$$
\delta W_{ss} = \int_{L} D_s(x) \delta v(x) \, dx \, + \, \int_{L} \left( \int_{\Gamma} \left[ \frac{\partial n_r}{\partial x} \right] T \frac{d^2 \delta v(x)}{dx^2} \, dx \right) \frac{d\delta v(x)}{dx} \, dx \, + \,
\int_{L} \left[ \int_{\Gamma} \left[ \sigma_s(x,y) - \frac{\partial}{\partial y} \left( \frac{\partial n_r}{\partial x} \right) \right] \frac{d^2 \delta v(x)}{dx^2} \, dx \right] \frac{d^2 \delta v(x)}{dx^2} \, dx \, + \,
\int_{L} \left[ \int_{\Gamma} \left[ M_{s}(x) - \int_{\Gamma} \sigma_s(x,y) \, dx \right] \frac{d^2 \delta v(x)}{dx^2} \, dx \right] \frac{d^2 \delta v(x)}{dx^2} \, dx
$$

In Eq. (16), the following sectional-moment contributions and sectional shear force can be defined:

$$
M_{\sigma_s}(x) = -\int_{\Gamma} y \sigma_s(x,y) \, dx; \\
M_{m_s}(x) = \int_{\Gamma} m_s(x) \, dx; \\
M_{r_s-t_s}(x) = -\int_{\Gamma} \left[ \frac{\partial}{\partial x} \left( \frac{\partial n_r}{\partial y} \right) \right] \left[ \frac{\partial n_r}{\partial x} \right] \, dx; \\
V_{r_s}(x) = \int_{\Gamma} n_r \, \frac{\partial n_r}{\partial x} \, dx,
$$

where $M_{\sigma_s}(x)$, $M_{m_s}(x)$, and $M_{r_s-t_s}(x)$ are the sectional moments contributed from the normal stress $\sigma_s(x,y)$ on the beam section, the couple stress $m_s(x)$ on the beam section, and the normal surface and residual surface stresses $r_s(x)$, $t_s(x)$ along the beam section perimeter, respectively; and $V_{r_s}(x)$ is the sectional shear force contributed from the transverse surface stress $n_r(x)$ along the beam section perimeter.

With Eqs. (17) and (18), the virtual work expression of Eq. (13) can be rewritten as:

$$
\int_{L} M(x) \frac{d^2 \delta v(x)}{dx^2} \, dx \, + \, \int_{L} V_{r_s}(x) \frac{d\delta v(x)}{dx} \, dx \, + \,
\int_{L} D_s(x) \delta v(x) \, dx \, - \, \int_{L} P_s(x) \delta v(x) \, dx \, - \, \delta \mathbf{U}^T \mathbf{P} = 0
$$

where $M(x) = M_{\sigma_s}(x) + M_{m_s}(x) + M_{r_s-t_s}(x)$ is the total section moment. It is clear from the first two terms in Eq. (19) that the total section moment $M(x)$ and the surface shear force $V_{r_s}(x)$ are conjugate-work pairs of the section curvature $\kappa(x) = \frac{d^2 v(x)}{dx^2}$ and the section rotation $\gamma(x) = \frac{dv(x)}{dx}$.

In order to move all differential operators to the bending moment $M(x)$ and the surface shear force $V_{r_s}(x)$, integration by parts is applied twice to the first term and once to the second term of Eq. (19), respectively, resulting in the following expression:

$$
\int_{L} \left( \frac{d^2 M(x)}{dx^2} \frac{dV_{r_s}(x)}{dx} + D_s(x) - P_s(x) \right) \delta v(x) \, dx \\
+ \int_{L} \left[ \frac{dM(x)}{dx} + V_{r_s}(x) \right] \frac{d\delta v(x)}{dx} \, dx \\
\, - \, \delta \mathbf{U}^T \mathbf{P} = 0
$$

The boundary terms in Eq. (20) reveal that the total sectional shear force $F(x)$ is not simply equal to the first derivative of the beam-section moment $M(x)$ like in the classical Euler-Bernoulli beam theory but is also contributed from the surface shear force $V_{r_s}(x)$. Thus, the total sectional shear force is defined as:

$$
F(x) = -\frac{dM(x)}{dx} + V_{r_s}(x).
$$

Following the Cartesian sign convention, Eq. (20) can be rewritten as:

$$
\int_{L} \left( \frac{d^2 M(x)}{dx^2} \frac{dV_{r_s}(x)}{dx} + D_s(x) - P_s(x) \right) \delta v(x) \, dx \\
\, - \, \delta U_i \left[ P_1 + \left( -\frac{dM(x)}{dx} + V_{r_s}(x) \right) \right]_{x=0} \\
\, - \, \delta U_2 \left[ P_2 + \left( M(x) \right) \right]_{x=L} \\
\, - \, \delta U_3 \left[ P_3 - \left( -\frac{dM(x)}{dx} + V_{r_s}(x) \right) \right]_{x=L} \\
\, - \, \delta U_4 \left[ P_4 - \left( M(x) \right) \right]_{x=L} = 0.
$$

Due to arbitrariness of $\delta v(x)$, the governing differential equilibrium equation of the beam-foundation system is obtained as:

$$
\frac{d^2 M(x)}{dx^2} \frac{dV_{r_s}(x)}{dx} + D_s(x) - P_s(x) = 0.
$$
Fig. 4. Tonti’s diagram for displacement-based beam on elastic substrate medium with inclusion of microstructure and surface effects.

Accounting for the arbitrariness of $\delta U$, the end-boundary force conditions (natural boundary conditions) are obtained as:

$$P_1 = \left[ -\frac{dM(x)}{dx} + V_{ta}(x) \right]_{x=0}; P_2 = \left( M(x) \right)_{x=0}$$

$$P_3 = \left( \frac{dM(x)}{dx} + V_{ta}(x) \right)_{x=L}; P_2 = \left( M(x) \right)_{x=L}. \quad (24)$$

It is noted that when compared to the governing differential equilibrium equation of the beam on Winkler-Pasternak foundation as given by Limkatanyu et al. [40], Eq. (23) and the one derived by Limkatanyu et al. [40] are the same. Thus, the problem of beams on elastic-substrate media with inclusion of microstructure and surface effects is equivalent to the problem of beams on Winkler-Pasternak foundation.

3.2 Sectional force-deformation relations

In order to establish the sectional constitutive relations, Eqs. (8) and (12) are substituted into Eqs. (17) and (18) as suggested by Gao and Mahmoud [36].

$$M(x) = (IE)_{sf} \kappa(x) \text{ and } V_{ta}(x) = (GA)_{sf} \gamma(x) \quad (25)$$

where the effective sectional flexural rigidity $(IE)_{sf}$ and the effective sectional shear rigidity $(GA)_{sf}$ are defined as:

$$(IE)_{sf} = \frac{IE(1-\nu)}{(1+\nu)(1-2\nu)} + (\lambda_0 + 2\mu_0)I_P + \mu I^2 A \quad (26)$$

$$(GA)_{sf} = \tau_0 S_P$$

with $A = \iint dA$ being the section area; $I = \iint y^2 dA$ being the second moment area; $I_P = \iint I \gamma^2 d\Gamma$ being the second moment perimeter; and $S_P = \iint n_x^2 d\Gamma$. It is noted in Eq. (26) that besides the microstructure and surface-energy effects, the effective sectional flexural rigidity $(IE)_{sf}$ also accounts for the effect of Poisson’s ratio.

The constitutive relation of the elastic-substrate spring is:

$$D_s(x) = k_s \Delta_s(x) \quad (27)$$

where $k_s$ is the elastic-substrate modulus known as subgrade-reaction coefficient [45].

It is noted that the governing differential equilibrium equation of Eq. (23), the end-force boundary conditions of Eq. (24), and the system constitutive relations of Eqs. (25) and (26) form a complete set of basic equations required for the displacement-based finite element formulation of the problem as summarized in the displacement-based Tonti’s diagram of Fig. 4 [46].
3.3 Differential compatibility equations and end compatibility conditions: the virtual force principle

The virtual force principle is an alternative way to express the system compatibility equations. The general form of the virtual force equation is written as:

\[ \delta W^* = \delta W'_{ae} + \delta W^*_{ae} = 0 \]  

(28)

where \( \delta W^* \) is the system total complementary virtual work; \( \delta W'_{ae} \) is the system internal complementary virtual work; and \( \delta W^*_{ae} \) is the system external complementary virtual work.

In the case of the beam on Winkler foundation with inclusion of microstructure and surface effects as shown in Fig. 3, \( \delta W'_{ae} \) and \( \delta W^*_{ae} \) can be expressed as:

\[
\delta W'_{ae} = \int_a^b \delta D_i(x) \Delta_i(x) \, dx + \int_a^b \delta D_t(x) \Delta_t(x) \, dx + \int_a^b \delta p \gamma(x) \, dx \\
\delta W^*_{ae} = -\int_a^b \delta P^T \gamma(x) - dx - \delta P^T U.
\]

(29)

(30)

Following the procedure employed by Limkatanyu et al. [40] and enforcing the governing differential equation of Eq. (23) to eliminate the elastic-substrate force \( D_i(x) \) and its virtual counterpart \( \delta D_i(x) \), the governing differential compatibility equations of the beam-section curvature and beam-section rotation are obtained as:

\[
\frac{M(x)}{(IE)_{eff}} + \frac{1}{k_s} \left( \frac{d^4 M(x)}{dx^4} - \frac{d^2 p_s(x)}{dx^2} \right) = 0
\]

(31)

\[
\frac{V_{te}(x)}{(GA)_{eff}} + \frac{1}{k_t} \left( \frac{d^3 V_{te}(x)}{dx^3} - \frac{d^2 V_{te}(x)}{dx^2} \right) = 0.
\]

(32)

Furthermore, accounting for the arbitrariness of \( \delta P \) yields the end-boundary compatibility conditions (essential boundary conditions):

\[
U_3 = \frac{1}{k_t} \left( \frac{d^2 V_{te}(x)}{dx^2} \right)_{x=L} + \frac{1}{k_t} \left( p_s(x) \right)_{x=L}
\]

\[
U_4 = \frac{1}{k_t} \left( \frac{d^2 V_{te}(x)}{dx^2} - \frac{d^2 M(x)}{dx^2} \right)_{x=L} + \frac{1}{k_t} \left( p_s(x) \right)_{x=L}.
\]

The governing differential compatibility equations of Eqs. (31) and (32) can be combined into a single expression as (see Ref. [40]):

\[
\frac{d^4 M(x)}{dx^4} + \lambda_1 M(x) - \lambda_2 \frac{d^2 M(x)}{dx^2} = \frac{d^2 p_s(x)}{dx^2}
\]

(34)

where \( \lambda_1 = k_s/((IE)_{eff}) \) and \( \lambda_2 = (GA)_{eff}/((IE)_{eff}) \). When compared to the governing differential equilibrium equation derived earlier, Eqs. (23) and (34) are dual. This confirms the duality of the virtual displacement and virtual force principles. As expected, the combined governing differential compatibility equation of Eq. (34) and the one given by Limkatanyu et al. [40] for the beam on Winkler-Pasternak foundation are in the same form. Furthermore, when the effects of microstructure, Poisson’s ratio, and surface energy are all neglected (\( t = v = \lambda_0 = \mu_0 = \tau_0 = 0 \)), Eq. (34) is reduced to the governing differential compatibility equation of the beam on Winkler foundation as given by Limkatanyu et al. [47].

It is noted that the governing differential compatibility equations of Eqs. (31) and (32), the end-boundary compatibility conditions of Eq. (33), and the system constitutive relations of Eqs. (25) and (26) form a complete set of basic equations required for the force-based finite element formulation of the problem as summarized in the force-based Tonli’s diagram of Fig. 5 [46].

4. “Exact” element stiffness matrix: reused

Due to similarity between the current problem and the one related to the beam on Winkler-Pasternak foundation, the “exact” element stiffness equation derived by Limkatanyu et al. [40] can be applied and is briefly discussed herein. The exact element stiffness matrix given by Limkatanyu et al. [40] is formulated based on the exact element flexibility matrix via the natural approach [48]. The matrix virtual force approach with the exact moment interpolation functions is employed to derive the exact element flexibility matrix. The analytical solution to the governing differential compatibility equation of Eq. (34) is central to obtain the exact moment interpolation functions. More details on the derivation of the exact element stiffness equation can be found in Limkatanyu et al. [40] and the element configuration of the beam element on Winkler-Pasternak foundation is shown in Fig. 6.

5. Numerical examples

Two numerical simulations employing the proposed model are performed to study the characteristics and behaviors of a beam-substrate system with inclusion of microstructure and surface effects. The first simulation involves investigation of the response of the beam resting on an elastic substrate. The second simulation examines the influences of several system parameters on contact stiffness and shows the size-dependent effect on the system response.
5.1 Example 1

A cantilever aluminum beam resting on an elastic substrate is subjected to a concentrated load \( P \) at its free end as shown in Fig. 7. Geometric properties of the aluminum beam follow those used by Gao and Mahmoud [36]. In all analysis cases, the beam cross-section shape is rectangular with a constant width-to-depth ratio \( b/h \) of 2 and the beam length \( L \) is kept to be 20\( h \). Bulk material and surface properties of the aluminum beam come from those used by Liu and Rajapakse [49] and Gao and Mahmoud [36]. The bulk modulus \( E \) and Poisson ratio \( \nu \) of the aluminum beam are 90 GPa and 0.23, respectively, while its residual surface stress \( \tau_0 \) is 0.5689 N/m and surface elastic constants \( k_0 \) and \( \mu_0 \) are 3.4939 and -5.4251 N/m, respectively. The length-scale parameter \( l \) for the bulk beam material (aluminum) is equal to 6.58 \( \mu \)m as given by Gao and Mahmoud [36]. Effects of the length-scale parameter \( l \) on the beam-deflection responses with different elastic-substrate stiffnesses \( k_s \) are investigated by varying the beam depth \( h \) as a function of \( l \). Thus, the effective sectional flexural rigidity \( (IE)_{sf} \), effective sectional shear rigidity \( (GA)_{sf} \), and beam

length \( L \) are also related to the value of \( l \).

For a rectangular beam section with width \( b \) and height \( h \), the sectional geometric properties are:

\[
A = bh; I = \frac{bh^3}{12}; I_p = \frac{k^3}{6} + \frac{bh^5}{2}; S_p = 2bh.
\]  

(35)

For convenience and generality, the following two non-dimensional variables are defined:

\[
\overline{k}_s = \frac{k_s L^4}{(IE)_{sf}} \quad \text{and} \quad \overline{P} = \frac{PL^2}{(IE)_{sf}}.
\]  

(36)

The first one reflects the substrate-stiffness effect while the second one normalizes different values of the applied load \( P \). In this numerical example, the value of the normalized load parameter \( \overline{P} \) is kept to be 1 while the normalized elastic-substrate stiffness parameter \( \overline{k}_s \) varies from 0.2 to 10.

Fig. 8 compares the beam deflection responses with different normalized substrate-stiffness parameters \( \overline{k}_s \) obtained
Fig. 7. Example I: cantilever beam on elastic substrate medium.

Fig. 8. Normalized beam deflection vs. normalized beam distance for various normalized elastic substrate stiffness.

with the proposed model and the classical beam model. The beam depth $h$ is expressed in terms of the length-scale parameter $l$ and varies from $l$ to $4l$. The classical beam response is simply obtained by neglecting the microstructure ($l = 0$) and surface-energy effects ($\lambda_0 = \mu_0 = r_0 = 0$). It is clear from Fig. 8 that when compared to the classical beam model, accounting for the microstructure and surface-energy effects consistently results in a stiffer beam-elastic substrate system. Fig. 8 also indicates that the beam deflection responses obtained with the proposed model and the classical beam model are significantly different when the beam depth $h$ approaches the length-scale parameter $l$ ($h = l = 6.58 \mu m$). However, this difference in the beam deflection responses decreases when the beam depth gets larger ($h = 4l = 26.32 \mu m$), especially with a stiff elastic substrate medium. Thus, the microstructure and surface-energy effects become dominant when the beam depth approaches the value of the material length-scale parameter, especially with a weak elastic substrate medium. This finding is in good agreement with that numerically observed by Park and Gao [4] and Gao and Mahmoud [36] and experimentally...
observed by McFarland and Colton [50].

5.2 Example II

Sensitivity of the model parameters on contact stiffness is investigated by performing parametric studies of the cantilever beam-substrate system in Fig. 7. The same beam bulk material and surface properties of the aluminum beam are employed in this example. The width-to-depth ratio $b/h$ is kept at 2 for all analysis cases. Model parameters investigated herein include the beam depth, the beam length, and the substrate stiffness. The slenderness ratio $L/h$ is used to define the beam-depth and beam-length effects. The substrate-stiffness effect is studied by varying the normalized elastic-substrate stiffness parameter $k_s$, from 0.2 to 50. Sensitivity analysis of model parameters on the contact stiffness is performed to measure essence of the microstructure and surface effects on the system response. Following the definition by Khajehansari et al. [38] and Jiang and Yan [51], the contact stiffness of a beam-substrate system is simply defined as:

$$K_{end} = \frac{P_{end}}{u_{end}}$$  \hspace{1cm} (37)

where $P_{end}$ and $u_{end}$ are the imposed force and the induced displacement at an end point, respectively.

In this study, two types of normalized contact stiffness are defined and used to assess the essence of the microstructure and surface effects on the system contact stiffness. The first normalized contact stiffness is used to represent the attribution of the microstructure effect and is defined as:

$$\bar{K}_{Micro} = \frac{K_{end, Micro}}{K_{end, Sur}}$$  \hspace{1cm} (38)

where $K_{end, Micro}$ is the contact stiffness accounting for both the microstructure and surface effect; and $K_{end, Sur}$ is the contact stiffness accounting for only the surface effect. The second normalized contact stiffness is employed to represent the ascription of the surface effect and is defined as:

$$\bar{K}_{Sur} = \frac{K_{end, Sur}}{K_{end, Micro}}$$  \hspace{1cm} (39)

where $K_{end, Micro}$ is the contact stiffness accounting for only the microstructure effect.

Figs. 9(a) and (b) shows influences of the beam length $L$ on the normalized microstructure contact stiffness $\bar{K}_{Micro}$ and the normalized surface contact stiffness $\bar{K}_{Sur}$ for beams resting on substrate media with different normalized substrate-stiffness parameter $k_s$, respectively. The beam length $L$ is varied by changing the slenderness ratio $L/h$ from 5 to 40 while the beam depth $h$ is retained equal to the microstructure length-scale parameter $l = 6.58 \mu m$. Fig. 9(a) illustrates that the beam length $L$ has a significant effect on the normalized microstructure contact stiffness $\bar{K}_{Micro}$, especially for lower values of normalized substrate-stiffness parameters. Thus, it can be deduced that the shorter the beam is and the softer the elastic substrate medium is, the larger the normalized microstructure contact stiffness $\bar{K}_{Micro}$ will be. Fig. 9(b) shows that the beam length $L$ practically has no effect on the normalized surface contact stiffness $\bar{K}_{Sur}$. Thus, it can be deduced that for these particular values of model parameters, influences of the microstructure effect are more pronounced than those of the surface effect. Furthermore, it is worth remarking that the surface effect would become more pronounced when the system dimension is in the order of nanometer. However, the system dimension investigated in this study is governed by the microstructure length-scale parameter $l = 6.58 \mu m$. Thus, the system dimension is in the order of micrometer.

Figs. 10(a) and (b) shows influences of the beam depth $h$ on the normalized microstructure contact stiffness $\bar{K}_{Micro}$ and the normalized surface contact stiffness $\bar{K}_{Sur}$ for beams resting on substrate media with different normalized substrate-stiffness parameter $k_s$, respectively. The beam depth $h$ is expressed in terms of the microstructure length-scale parameter $l$ while the beam length $L$ is retained at 25 $h$. Fig. 10(a) shows that the microstructure size effect is significant when the beam depth $h$ approaches the microstructure length-scale parameter $l$, especially for lower values of substrate-stiffness parameters and diminishes when the beam depth $h$ approaches a threshold value around $6l$. Similar to the observation for Fig. 9(b), Fig. 10(b) indicates that the beam depth $h$ has no effect...
Similarity between the current problem and the one related to the beam on Winkler-Pasternak foundation is observed. Consequently, the "natural" Winkler-Pasternak-based beam element previously proposed by the authors can be reused to study the problem of beams resting on elastic substrate media with inclusion of microstructure and surface effects. Two numerical simulations are performed to study characteristics and behaviors of the micro-sized beam-substrate system.

The first simulation indicates that accounting for the microstructure and surface-energy effects consistently results in a stiffer beam-elastic substrate system in the same way as increasing the beam flexural rigidity when compared to the classical beam model. The beam deflection responses obtained with the proposed model and the classical beam model are significantly different when the beam depth \( h \) approaches the length-scale parameter \( l \) (\( h = l = 6.58 \mu m \)). However, this difference in the beam deflection responses decreases when the beam depth gets larger (\( h = 4l = 26.32 \mu m \)), especially with a stiffer elastic substrate medium.

The second simulation points out that influences of the microstructure effect are more pronounced than those of the surface effect when the system dimension is in the order of micrometer. A stiff elastic substrate medium tends to diminish the size-dependent characteristic of the beam-substrate medium system.

One next step in this research direction is to include nonlinearities into both the beam and the substrate medium. It is anticipated that the beam-substrate medium element developed herein will be useful to scientists and engineers working in the area of nanoscience and nanotechnology.

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